A COURSE IN GEOMETRY

ELEMENTARY ANALYTICAL GEOMETRY

By T. H. Ward Hill, M.A. Crown 8vo, 272 pages, **4s. 6d.**

GEOMETRY

For Secondary Schools. By J. C. Pincock, M.A. Crown 8vo, 320 pages, **3s. 6d.** Also in Two Parts, **2s.** each.

THE ELEMENTS OF PLANE TRIGONOMETRY

By J. T. Brown, M.A., B.Sc., and A. Martin, M.A., B.Sc. Crown 8vo, 190 pages, **3s. 6d.** Also with Answers, **3s. 6d.**

A COMPLETE SCHOOL ALGEBRA

By A. Ritchie-Scott, D.Sc., F.R.S.E. Crown 8vo, 646 pages, **7s. 6d.**; with Answers, **8s. 6d.** Also in two Parts, **4s. 6d.** each; with Answers, **5s.** each.

A COURSE IN GEOMETRY

BY

J. L. LATIMER M.A.
FORMERLY SCHOLAR OF ST JOHN'S COLLEGE OXFORD
ASSISTANT MASTER MANCHESTER GRAMMAR SCHOOL

AND

T. SMITH B.Sc.
ASSISTANT MASTER MANCHESTER GRAMMAR SCHOOL

GEORGE G. HARRAP & COMPANY LTD.
LONDON BOMBAY SYDNEY

First published 1937
by George G. Harrap & Co. Ltd.
182 *High Holborn, London, W.C.*1

Copyright. All rights reserved

MADE IN GREAT BRITAIN. PRINTED AT THE PITMAN PRESS, BATH

PREFACE

In this book we have attempted to employ a style in setting out propositions and riders which leaves the pupil free to concentrate on the purely mathematical difficulties of the subject. Pupils may be taught this technique in the earliest days of their 'B' or deductive course. The methods of setting out propositions and illustrative examples are those employed by the mathematical staff of the Manchester Grammar School. The use of single letters for angles has been practised at this school for some years, and we are convinced that the time saved in writing out work and the ease with which such work may be read more than compensate for the necessary 'drill' required to ensure large, clear figures in which the angles are plainly and legibly marked by single letters. Without this drill figures tend to become confused and illegible.

The exercises are numerous and varied. Most groups of exercises are divided into two sections, A and B. Section A will usually be sufficient for the average pupil in his first reading. This will leave most of Section B for revision, and the end questions of this section for the better pupils. Three-dimensional work is not treated formally, but some of the examples demand an intuitive knowledge of the properties of solid figures. In addition there are four graded sets of revision papers, which come after the usual first, second, third, and fourth year's work in secondary schools. Again a few of the papers at the end of each set are intended for the more able pupils.

The Preliminary Course consists of examples only, all explanation being left to the teacher. The remainder of the book is a straightforward course up to School Certificate standard, together with a very short course beyond this standard. It does not profess to replace the teacher; but it is hoped that the teacher need not hesitate to put the book

into a pupil's hand either for preliminary reading or revision. The introductory work in connexion with limits is left entirely to the teacher's discretion.

We have arranged the constructions in four main groups, but it is not intended that they should be taken together or as they occur in the text. Some examination authorities still require 'proofs' of congruence theorems: these are given at the end of the book. On pp. 11–32 is given a "Summary of Theorems and Constructions," with diagrams, which we hope will prove useful for revision purposes. Alternative proofs are provided for some theorems and constructions; a few blank pages are supplied at the end for recording other alternatives or notes.

We are indebted to the members of the mathematical staff of the Manchester Grammar School for much valuable criticism.

J. L. L.
T. S.

CONTENTS

PAGE

Summary of Theorems and Constructions 11

PRELIMINARY COURSE

Measurement of Lines and Angles 33
Scale Drawing 36
Angles of a Triangle 37
Construction of Triangles and Quadrilaterals 38
Construction of Solid Figures 40
Scale Drawing: Three-dimensional Problems 44
Constructions 47
Locus 49

PART I

FUNDAMENTAL IDEAS

Solids, Surfaces, Lines, Points, and Angles 52
Facts about Parallels 60
Definitions 68

PART II

GEOMETRICAL THEOREMS

Congruent Triangles 78
Isosceles and Equilateral Triangles 84
Inequalities 90
Parallelograms 98
Converse Theorems 106
Intercept Theorems 110
Angles of a Polygon 115
Symmetry 119

PAGE
Constructions 120
Test Papers I–XXX 142

PART III

AREAS

Areas 153
Constructions 174
Locus 180
Test Papers XXXI–LIV 193

PART IV

THE CIRCLE

Chords of a Circle 202
Arc, Segment, Angle in a Segment 209
Concyclic Points and Cyclic Quadrilateral 214
Arcs 229
Tangent and Secant 232
Constructions 253
Test Papers LV–LXXVI 272

PART V

FURTHER AREAS AND RATIO

Algebraic Identities 279
Projection 284
Construction 294
Ratio 296
Proportion 296
Similar Figures 311
Segments of Chords of a Circle 324
Areas of Similar Triangles 332
Constructions 336
Test Papers LXXVII–C 347

PART VI

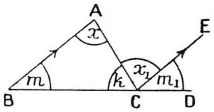
MISCELLANEOUS THEOREMS

	PAGE
Directed Lengths	356
Appendix: Congruence Theorems	370
Answers	379

SUMMARY OF THEOREMS AND CONSTRUCTIONS

THEOREM 1

If one side of a triangle be produced the exterior angle so formed is equal to the sum of the two opposite interior angles of the triangle. The sum of the three angles of the triangle is two right angles.

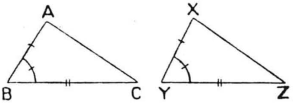


Prove that

(i) $\widehat{ACD} = x + m$.
(ii) $x + m + k = 180°$.

THEOREM 2

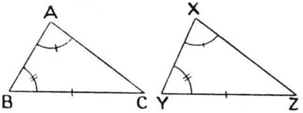
If two triangles have two sides of the one respectively equal to two sides of the other and the included angle of the one equal to the included angle of the other, then the triangles are congruent.



Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

THEOREM 3

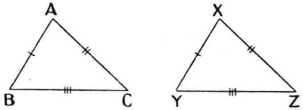
If two triangles have two angles of the one respectively equal to two angles of the other and a side of the one equal to the corresponding side of the other, then the triangles are congruent.



Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

THEOREM 6

If two triangles have three sides of the one respectively equal to three sides of the other, then the triangles are congruent.

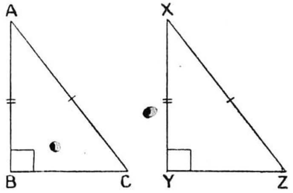


Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

Theorem 7

If two right-angled triangles have their hypotenuses equal and another side of one triangle equal to a second side of the other triangle, then the triangles are congruent.

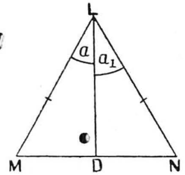
Prove that △ ABC ≡ △ XYZ.



Theorem 4

If two sides of a triangle are equal, then the angles opposite these sides are equal.

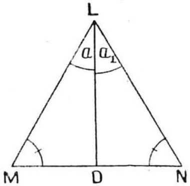
Prove that $\hat{M} = \hat{N}$.



Theorem 5

If in a triangle two angles are equal, then the sides opposite these angles are also equal.

Prove that LM = LN.

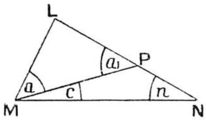


Theorem 8

If in a triangle one side is greater than a second side, then the angle opposite the greater side is greater than the angle opposite the smaller side.

Prove that if LN > LM,

then $\widehat{LMN} > \widehat{LNM}$.

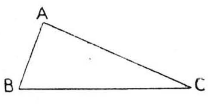


Theorem 9

If in a triangle one angle be greater than a second angle, then the side opposite the greater angle is greater than the side opposite the smaller angle.

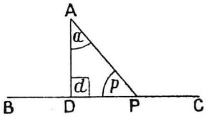
Prove that if $\hat{B} > \hat{C}$,

then AC > AB.



Theorem 10

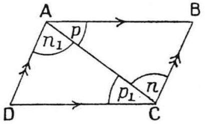
If a point be taken outside a given straight line, then of all the lines that may be drawn from this point to the given line the perpendicular is the shortest.



Prove that AD is less than AP.

Theorem 11

In a parallelogram the opposite sides are equal and the opposite angles are equal.



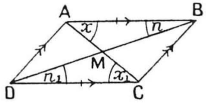
Prove that

(i) AB = DC and BC = AD.

(ii) $\hat{B} = \hat{D}$ and $B\widehat{A}D = D\widehat{C}B$.

Theorem 12

In a parallelogram the diagonals bisect each other.

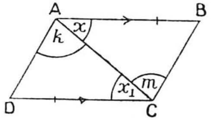


Prove that AM = CM

and BM = DM.

Theorem 13

If one pair of opposite sides of a quadrilateral are equal and parallel, then the second pair of opposite sides are equal and parallel.

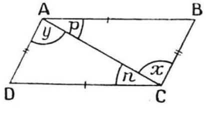


Prove that BC = DA

and BC ∥ DA.

Theorem 14

If the opposite sides of a quadrilateral are equal the quadrilateral is a parallelogram.

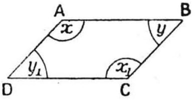


Prove that ABCD is a parallelogram.

Theorem 15

If the opposite angles of a quadrilateral are equal the quadrilateral is a parallelogram.

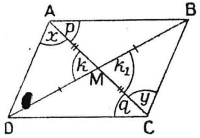
Prove that ABCD is a parallelogram.



Theorem 16

If the diagonals of a quadrilateral bisect each other the quadrilateral is a parallelogram.

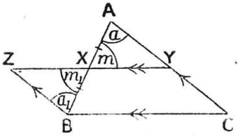
Prove that ABCD is a parallelogram.



Theorem 17

If in any triangle a line be drawn through the mid-point of one side parallel to a second side this line will bisect the third side.

Prove that AY = CY.

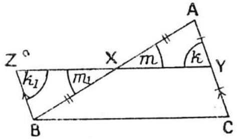


Theorem 18

The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.

Prove that XY || BC

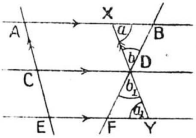
and XY = $\frac{1}{2}$BC.



Theorem 19

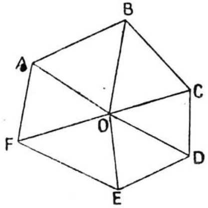
If three or more parallel straight lines cut off equal intercepts on one straight line which cuts them, then they cut off equal intercepts on any other straight line which cuts them.

Prove that BD = FD.



Theorem 20

In a convex polygon of n sides the sum of the interior angles is $(2n - 4)$ right angles.

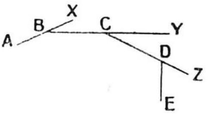


Prove that

$\widehat{ABC} + \widehat{BCD} + \ldots = (2n - 4)$ right angles.

Theorem 21

If at each vertex of a convex polygon of n sides an exterior angle be formed the sum of these exterior angles is 4 right angles.

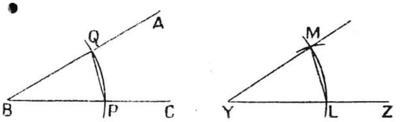


Prove that

$\widehat{XBC} + \widehat{YCD} + \widehat{ZDE} \ldots = 4$ right angles.

Construction 1

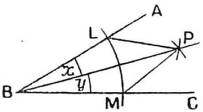
To construct a line which passes through a given point in a given straight line and makes with that line an angle equal to a given angle.



Construct $\hat{Y} = \hat{B}$.

Construction 2

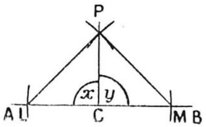
To bisect a given angle.



Bisect $\widehat{ABC}$.

Construction 3

To construct a straight line at right angles to a given unlimited straight line from a given point on the line.

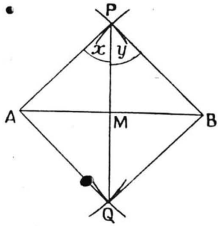


Construct CP perpendicular to AB.

Construction 4

To construct a straight line to bisect a given limited straight line.

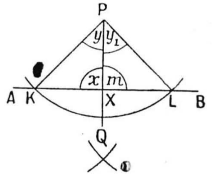
Construct PQ to bisect AB.



Construction 5

To construct a perpendicular to an unlimited straight line from a point outside the line.

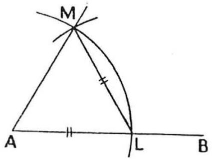
Construct PX perpendicular to AB.



Construction 6

To construct an angle of 60° without using a protractor.

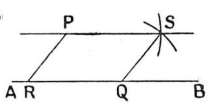
Construct $\widehat{A} = 60°$.



Construction 7

To construct through a given point a straight line parallel to a given straight line.

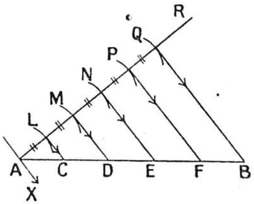
Construct PS || AB.



Construction 8

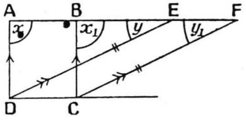
To show how to divide a given straight line into a given number of equal parts.

Divide AB into five equal parts.



THEOREM 22

If a parallelogram and a rectangle have the same base and lie between the same parallels they are equal in area.

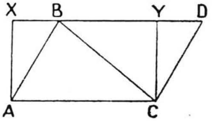


Prove that

rect. ABCD = par^m CDEF in area.

THEOREM 23

The area of a triangle is equal to half the area of a rectangle drawn on the same base and between the same parallels.

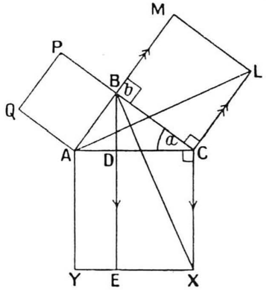


Prove that

$\triangle$ ABC $= \frac{1}{2}$ rect. ACYX in area.

THEOREM 24

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.

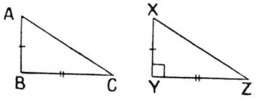


Prove that

sq. AX = sq. BL + sq. BQ in area.

THEOREM 25

If the square on one side of a triangle is equal to the sum of the squares on the other two sides the triangle is right-angled.

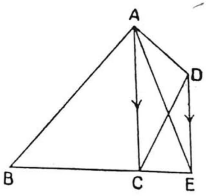


Prove that if $AC^2 = AB^2 + BC^2$,

then $\widehat{B}$ = a right angle.

CONSTRUCTION 9

To construct a triangle equal in area to a given quadrilateral.

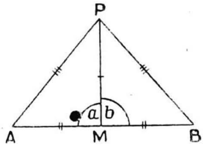


Construct

$\triangle$ ABE = quad^l ABCD in area.

Theorem 26 (i)

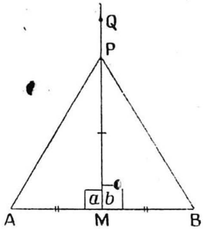
A point which is equidistant from two given points lies on the perpendicular bisector of the line joining the given points.



Prove that P lies on the perp^r bisector of AB.

Theorem 26 (ii)

Any point on the perpendicular bisector of a straight line joining two given points is equidistant from the given points.



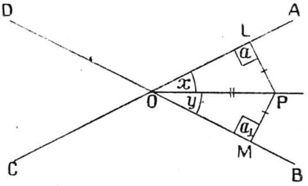
Prove that PA = PB.

Theorem 26 (iii)

The locus of a point which moves so as to be equidistant from two given points is the perpendicular bisector of the line joining the given points.

Theorem 27 (i)

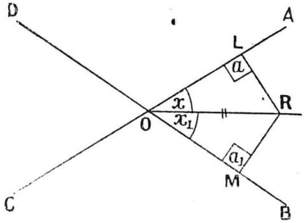
A point which moves within an angle formed by two intersecting straight lines and is equidistant from the lines moves on the bisector of the angle.



Prove that $x = y$.

Theorem 27 (ii)

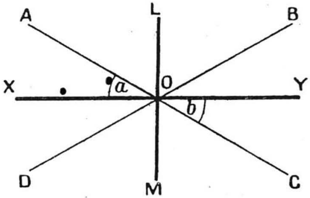
Any point on the bisector of a given angle is equidistant from the arms of the angle.



Prove that RL = RM.

Theorem 27 (iii)

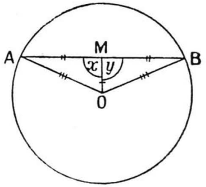
The locus of points which are equidistant from two intersecting straight lines is the pair of straight lines which bisect the angles between the given lines.



Theorem 28

A straight line drawn from the centre of a circle to bisect a given chord which is not a diameter is at right angles to the chord.

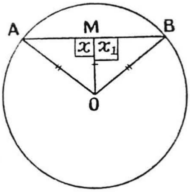
Prove that $x = 90°$.



Theorem 29

If a line be drawn from the centre of a circle perpendicular to a chord of the circle it will bisect the chord.

Prove that AM = MB.



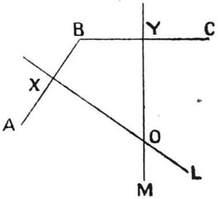
Theorem 30

There is one circle, and only one, which passes through three points not in the same straight line.

Prove that

(i) A circle can be drawn to pass through A, B, and C.

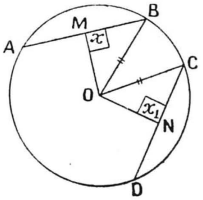
(ii) Only one such circle can be drawn.



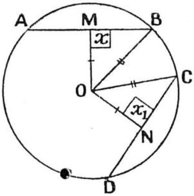
Theorem 31

Equal chords of a circle are equidistant from the centre.

Prove that if AB = CD,
then OM = ON.



Theorem 32

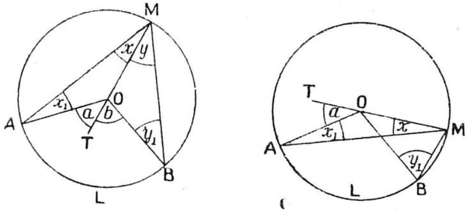


Chords which are equidistant from the centre of a circle are equal.

Prove that if OM = ON,
then AB = CD.

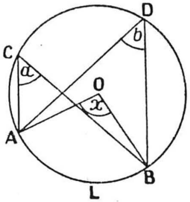
Theorem 33

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.



Prove that $\widehat{AOB} = 2\widehat{AMB}$.

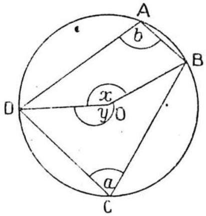
Theorem 34



Angles in the same segment of a circle are equal.

Prove that $a = b$.

Theorem 35 (i)



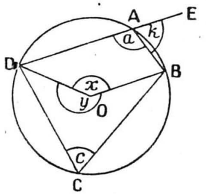
The opposite angles of a quadrilateral inscribed in a circle are supplementary.

Prove that $a + b = 180°$.

THEOREM 35 (ii)

If one side of a quadrilateral inscribed in a circle is produced the exterior angle so formed is equal to the opposite interior angle of the quadrilateral.

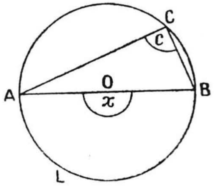
Prove that $k = c$.



THEOREM 36

The angle in a semicircle is a right angle.

Prove that $c = 90°$.

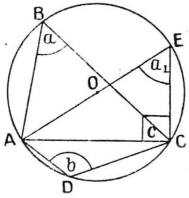


THEOREM 37

Any angle in a segment of a circle less than a semicircle is greater than one right angle, and any angle in a segment greater than a semicircle is less than one right angle.

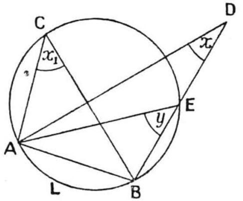
Prove that

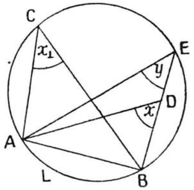
(i) $a < 90°$.
(ii) $b > 90°$.



THEOREM 38

If the line joining two points subtends equal angles at two other points on the same side of it the four points lie on a circle.

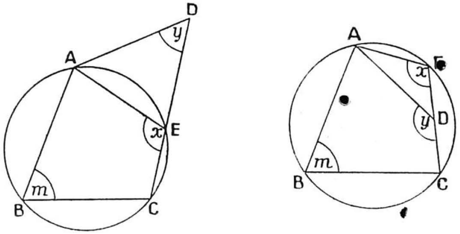




Prove that A, B, C, and D lie on a circle.

THEOREM 39

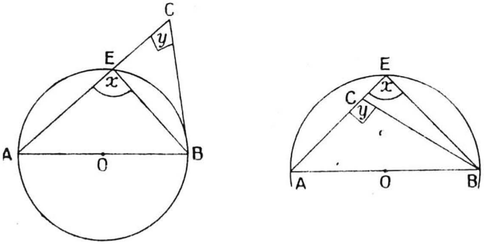
A quadrilateral in which two opposite angles are supplementary is cyclic.



Prove that if $m + y = 180°$,
then A, B, C, and D lie on a circle.

THEOREM 40

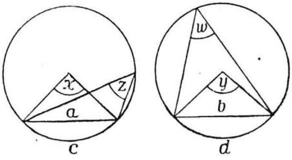
If the line joining two points subtends a right angle at a third point, then this point will lie on the circle on the given line as diameter.



Prove that C lies on the circle on AB as diameter.

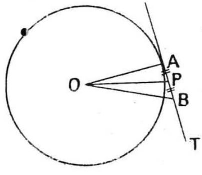
THEOREM 41

Theorems connecting arcs, chords, and angles in equal circles are comprised under this heading.



THEOREM 42

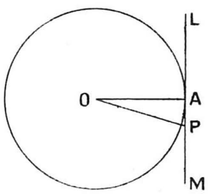
The tangent to a circle at any point on its circumference is perpendicular to the radius of the circle which passes through the point of contact.



Prove that, if AT is a tangent, then $\widehat{OAT} = 90°$.

THEOREM 43

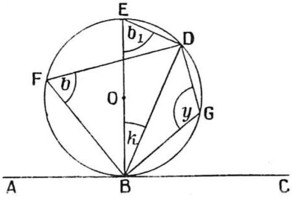
The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.



Prove that if $\widehat{OAM} = 90°$, then LAM is a tangent to the circle.

THEOREM 44

The angle between a tangent to a circle and a chord through the point of contact is equal to any angle in the alternate segment cut off by the chord.



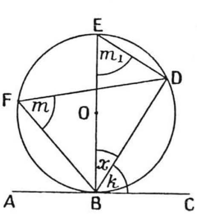
Prove that

(i) $\widehat{DBC} = b$.

(ii) $\widehat{DBA} = y$.

THEOREM 45

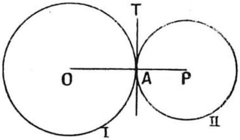
If a straight line be drawn from the extremity of a chord of a circle making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.

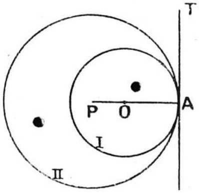


Prove that ABC is a tangent to the circle.

THEOREM 46

If two circles touch the point of contact lies on the straight line through the centres.



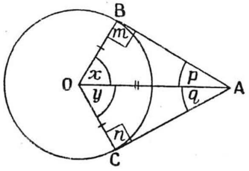


Prove that OAP is a straight line.

THEOREM 47

If two tangents are drawn to a circle from an external point, then

(i) they are equal in length;
(ii) the angle between them is bisected by the line joining the external point to the centre of the circle;
(iii) the same line bisects the angle between the radii drawn to the points of contact.

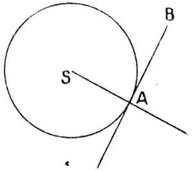


Prove that

(i) AB = AC.
(ii) $p = q$.
(iii) $x = y$.

CONSTRUCTION 10

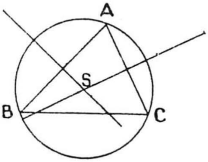
To construct a tangent to a circle through a given point on the circle.



Construct the tangent at A to the circle.

CONSTRUCTION 11

To construct the circum-circle of a given triangle.

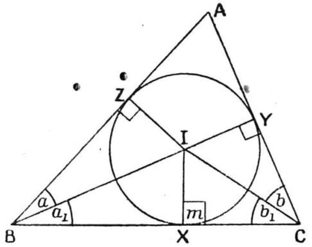


Construct the circum-circle of the △ ABC.

Construction 12

To construct a circle to touch the three sides of a given triangle internally.

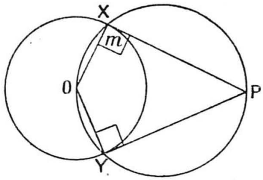
Construct a circle which will touch AB, BC, and CA internally.



Construction 13

To construct two tangents to a circle from a given external point.

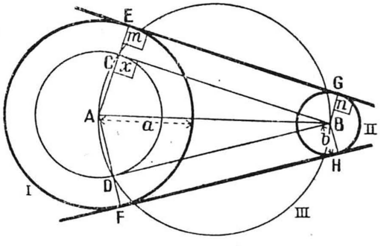
Construct two tangents from P to the circle.



Construction 14

To construct two direct common tangents to two given unequal circles.

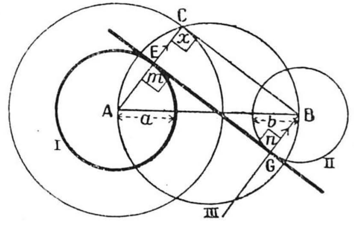
Construct two direct common tangents to the circles I and II.



Construction 15

To construct a transverse common tangent to two given circles.

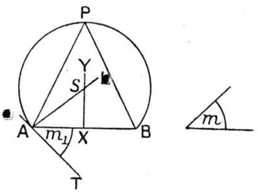
Construct a tranverse common tangent.



CONSTRUCTION 16

To construct the locus of a variable point at which a given fixed straight line subtends a given angle, if the moving point lies on one side only of the given straight line.

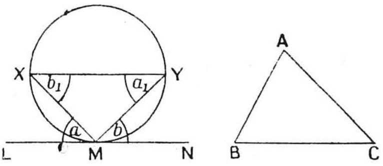
Construct the locus of a variable point at which AB subtends an angle equal to m.



CONSTRUCTION 17

To construct in a given circle a triangle whose angles shall be equal to those of a given triangle.

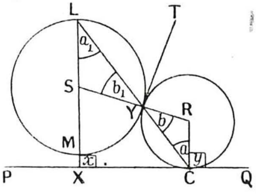
Construct a triangle in the circle equiangular to $\triangle$ ABC.



CONSTRUCTION 18

To construct a circle to touch a given circle and to touch a given straight line at a given point.

Construct a circle to touch the given circle and to touch PQ at C.

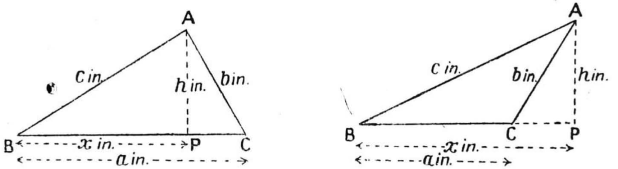


To illustrate the algebraic identities:

A. $k(a + b + c) \equiv ka + kb + kc.$
B. $(a + b)^2 \equiv a^2 + 2ab + b^2.$
C. $(a - b)^2 \equiv a^2 - 2ab + b^2.$
D. $a^2 - b^2 \equiv (a + b)(a - b).$

Theorem 48

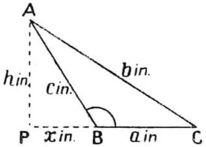
In any triangle the square on the side opposite to an ***acute*** angle is equal to the sum of the squares on the other two sides ***minus*** twice the rectangle contained by one of those sides and the projection of the other upon it.



Prove that $b^2 = a^2 + c^2 - 2ax.$

Theorem 49

In any obtuse-angled triangle the square on the side opposite to an ***obtuse*** angle is equal to the sum of the squares on the other two sides ***plus*** twice the rectangle contained by one of those sides and the projection of the other one upon it.

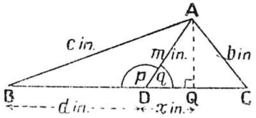


Prove that $b^2 = c^2 + a^2 + 2ax.$

Theorem 50

(Theorem of Apollonius)

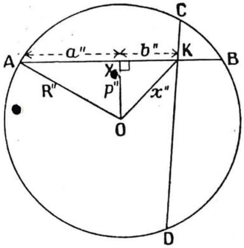
In any triangle the sum of the squares on two sides is equal to twice the square on half the third side plus twice the square on the median bisecting this third side.



Prove that $b^2 + c^2 = 2d^2 + 2m^2.$

THEOREM 51

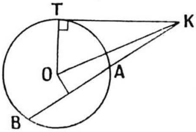
If two chords of a circle intersect within the circle the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other chord; each rectangle is equal to the square on the radius diminished by the square on the line joining the point of intersection of the chords to the centre of the circle.



Prove that
AK.KB = CK.KD = $R^2 - OK^2$.

THEOREM 52

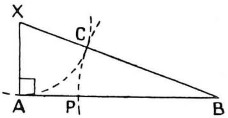
If from a point outside a circle a tangent and a secant be drawn, then the square on the tangent is equal to the rectangle contained by the secant and that part of the secant which lies outside the circle.



Prove that KT^2 = KA.KB.

CONSTRUCTION 19

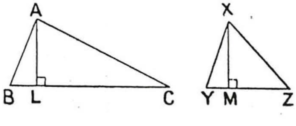
To divide a straight line into two parts such that the rectangle contained by the whole line and one part shall be equal to the square on the other part.



Construct a point P on AB such that AP.AB = PB^2.

THEOREM 53

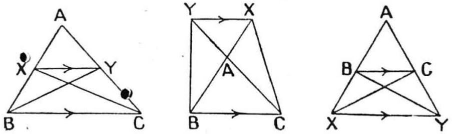
The ratio of the areas of triangles with equal altitudes is equal to the ratio of their bases.



Prove that $\frac{\triangle ABC}{\triangle XYZ} = \frac{BC}{YZ}$.

THEOREM 54

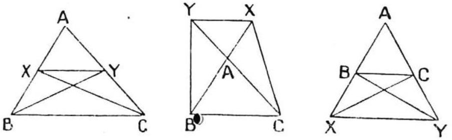
If a straight line is drawn parallel to one side of a triangle it cuts the other two sides proportionally.



Prove that $\frac{AX}{XB} = \frac{AY}{YC}$.

THEOREM 55

A straight line which cuts two sides of a triangle proportionally is parallel to the third side of the triangle.

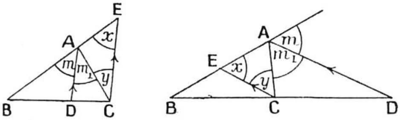


Prove that if $\frac{AX}{XB} = \frac{AY}{YC}$,

then BC $\parallel$ XY.

THEOREM 56

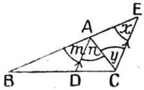
If the interior or exterior vertical angle of a triangle is bisected the bisector divides the opposite side in the ratio of the sides containing the bisected angle.

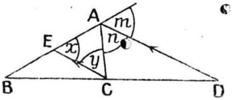


Prove that $\frac{BD}{DC} = \frac{BA}{AC}$.

THEOREM 57

If a line is drawn through the vertex of a triangle to divide the base, internally and externally, in the ratio of the other two sides of the triangle, then this line bisects the vertical angle of the triangle internally or externally.





Prove that if $\frac{BD}{DC} = \frac{BA}{AC}$,

then $m = n$.

THEOREM 58

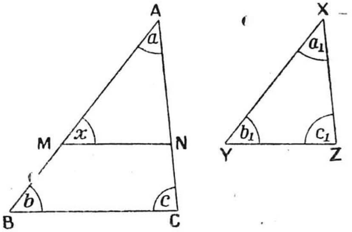
If two triangles are equiangular the triangles are similar.

Prove that if

$a = a_1$, $b = b_1$, and $c = c_1$,

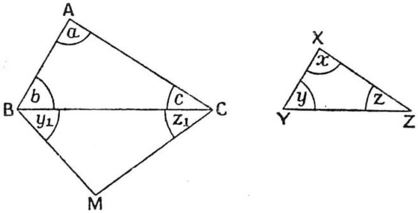
then

$\triangle$s ABC and XYZ are similar.



THEOREM 59

If the three sides of one triangle are proportional to the three sides of a second triangle the triangles are similar.



Prove that if $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$,

then $\triangle$s ABC and XYZ are similar.

THEOREM 60

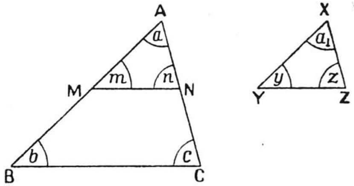
If two triangles have an angle of the one equal to an angle of the other and the sides about these equal angles in proportion, then the triangles are similar.

Prove that if

$$a = a_1 \text{ and } \frac{AB}{XY} = \frac{AC}{XZ},$$

then

$\triangle$s ABC and XYZ are similar.



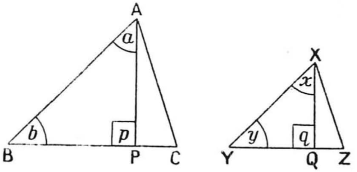
THEOREMS 51, 52

The consideration of Theorems 51 and 52, worked by a second method, and their converses, enters at this point.

THEOREM 61

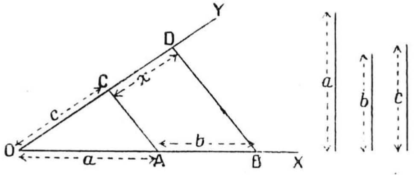
The ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

Prove that $\frac{\Delta_1}{\Delta_2} = \frac{BC^2}{YZ^2}$.



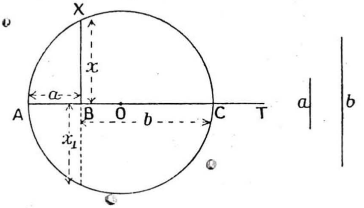
CONSTRUCTION 20

To construct a fourth proportional to three given lines.



Construct a line x such that $a : b = c : x$.

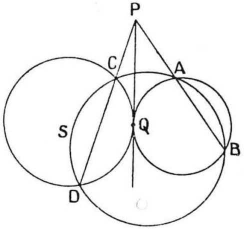
Construction 21



To construct a mean proportional to two given lines.

Construct a mean proportional to a and b.

Construction 22

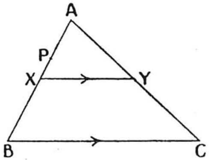


To construct a circle which shall pass through two given points and touch a given circle.

Construct a circle to pass through A and B and to touch the given circle.

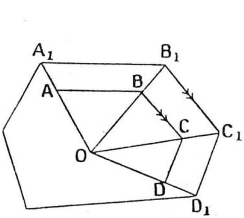
Construction 23

To cut off from a given triangle, by a line drawn parallel to one side, a triangle whose area shall bear to the area of the first triangle a given ratio.



Construct XY parallel to BC such that $\frac{\triangle AXY}{\triangle ABC} = \frac{3}{7}$.

Construction 24



To construct a polygon similar to a given polygon with an area equal to a given fraction of the area of the given polygon.

Construct a polygon ABCD . . . similar to the polygon $A_1B_1C_1D_1$ . . . and such that the ratio of their areas is $\frac{5}{7}$.

PRELIMINARY COURSE

MEASUREMENT OF LINES AND ANGLES

EXERCISES: I

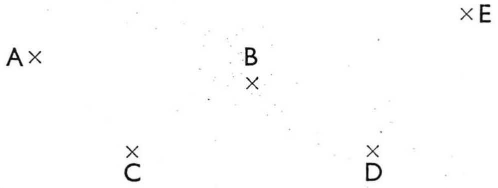


FIG. 1

LINES	LENGTHS, IN INCHES	LENGTHS, IN CENTIMETRES
AB		
BE		
CE		
DB		
CD		
AD		
BC		

1. Copy the table above, and complete by measuring the lengths of the lines obtainable from Fig. 1 (*a*) to two decimal places of an inch, (*b*) to one decimal place of a centimetre, and inserting these lengths in the appropriate places in the table.

2. Measure the length of a page of this book in inches and in centimetres.

3. Guess which is longer, AB or CD in Fig. 2. Test by measuring in centimetres.

FIG. 2

4. Draw any straight line AB. Make a mark at a point C which you guess to be the mid-point of AB. Test by measuring AC and CB.

5. Guess which is longer, BC or CD in Fig. 3. Test by measuring.

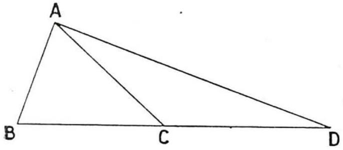


FIG. 3

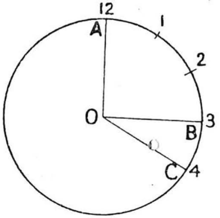


FIG. 4

Consider the diagram of a clock shown in Fig. 4. In turning from 12 o'clock to 20 minutes past 12 the minute hand is said to turn through the angle AOC, which we write for short as $\widehat{AOC}$.

The angle AOB in the figure—*i.e.*, the angle between the lines OA and OB—is called a right angle, or 90° (90 degrees).

The angle turned through by the hour hand between 12 o'clock and 1 o'clock is one-third of a right angle, or 30°.

Angles are measured with a *protractor*.

Fig. 5 shows the chief compass points.

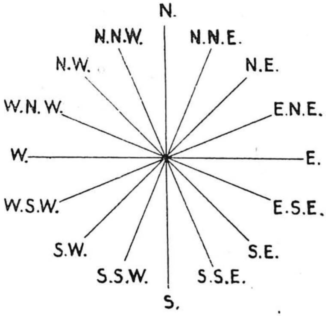


FIG. 5

The direction S. 67° E. means 67° east of south.

If a straight line OP revolves about O in a clockwise

direction from a position OA (A being north of O) to a position OB, then the angle through which OP turns is said to be the true bearing of B as seen from O.

Thus, the true bearing of the direction S.W. is 225° (Fig. 6).

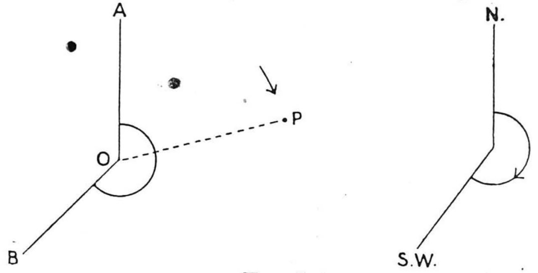


FIG. 6

EXERCISES: II

1. Name and measure the angles in Fig. 7.

2. Draw any *triangle*. Measure the three angles of the triangle. What is their sum?

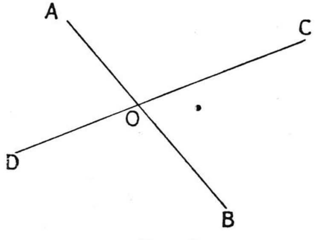


FIG. 7

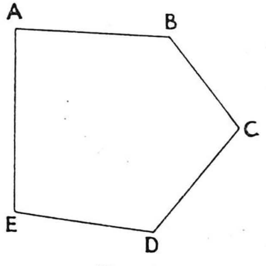


FIG. 8

3. Name and measure the angles in Fig. 8. What is the sum of the five angles?

4. Draw what you consider to be an angle of 50°. Test by measuring.

5. Draw what you consider to be a line *bisecting* the angle drawn for the previous exercise. Test by measuring the two parts.

6. Draw what you consider to be an angle of 160°, and draw two lines which *trisect* it. Test by measurement.

(It is suggested that the diagrams throughout the book be used for extra exercises in guessing and then measuring the sizes of angles.)

7. Calculate the angle between the positions of the hour hand of a clock:

(*a*) At 4 o'clock and 8 o'clock.
(*b*) At 3 o'clock and 5 o'clock.
(*c*) At 7 o'clock and half-past 8.

8. Calculate the angle between the positions of the minute hand of a clock:

(*a*) At 2.10 and 2.15.
(*b*) At 2.5 and 2.30.
(*c*) At 2.5 and 2.55.
(*d*) At 2.5 and 2.17.

9. Find the angle between the following directions:

(*a*) N.E. and S.E.
(*b*) W. and W.N.W.
(*c*) W.S.W. and S.E.
(*d*) N. 15° W. and N. 35° E.
(*e*) N. 50° W. and S. 80° E.

10. Give the true bearing of the following directions:

(*a*) N.E.
(*b*) N. 47° E.
(*c*) S. 80° E.
(*d*) N.W.
(*e*) S. 50° W.

SCALE DRAWING

Exercises: III

1. Find the length of the *diagonal* of a page of a book which is 10 in. long and 8 in. wide.

2. A ship sails 4 miles due east and then 5 miles due south. How far is it from its starting-point? (Draw a diagram with 1 in. representing 1 mile.)

3. A ship sails 3 miles due east, then 3 miles due south, and finally 1 mile due west. How far is it from its starting-point?

4. A ship sails 4 miles due west and then 3 miles in a direction N. 50° E. How far is the ship from its starting-point? What is the *bearing* of the ship from its starting-point?

5. A ship sails 7 miles due north and then 11 miles in a direction N. 70° W. How far is the ship from its starting-point? What is the bearing of the ship from its starting-point? (Scale: 1 cm. represents 1 mile.)

6. A ship sails 2 miles in a direction N. 40° E. and then 3 miles in a direction N. 70° W. How far is the ship from its starting-point? What is the bearing of the ship from its starting-point?

7. A ladder placed with its base 4 ft. from the wall of a house makes an angle of 70° with the ground. How long is the ladder?

8. A man 40 ft. from the wall of a house finds the *angle of elevation* of the top of the wall to be 50°. How high is the wall? (Scale: 1 in. represents 10 ft.)

9. What would be the height of the wall in the preceding question if the man were 30 ft. from it and the angle of elevation were 40°?

10. The string of a kite is 120 ft. long and makes an angle of 64° with the ground. How high is the kite above the ground?

11. From the top of a *vertical* cliff 300 ft. high the *angle of depression* of a buoy out at sea is 22°. How far is the buoy (*a*) from the base of the cliff, (*b*) from the top of the cliff?

12. A buoy is known to be 200 yds. from the base of a vertical cliff. From the top of the cliff its angle of depression is 31°. How high is the cliff?

13. A boat is 600 ft. from the base of a cliff 250 ft. high. What is its angle of depression from the top of the cliff?

14. From a point on the ground the angle of elevation of the top of a tower is 23°. From a point on the ground 40 ft. nearer the tower the angle of elevation is 39°. How high is the tower?

15. From a point on the ground 40 ft. from the base of a tower the angle of elevation of the top of the tower is 48° and the angle of elevation of the top of a flagstaff on the tower is 61°. How high is the flagstaff?

16. A man 6 ft. high stands with his toes 12 ft. from the bottom of a street lamp. His shadow is 18 ft. long. How high is the lamp?

ANGLES OF A TRIANGLE

Exercises: IV

1. Draw any triangle. Measure the three angles and write down their sum.

2. Draw on a sheet of paper any triangle. Cut it out. Cut off the three corners. Place these in *adjacent* positions. Deduce the sum of the three angles of the triangle.

3. Deduce the sum of the four angles of a *quadrilateral* by drawing a diagonal.

4. In Fig. 9 what is the value of $a + b + c + d + e$? (Note that this sum is that of the angles of a *pentagon*.)

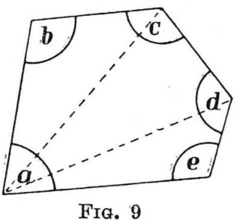


Fig. 9

(For the following examples the results of the preceding three exercises will be needed.)

5. Find the third angle of a triangle if the other two angles are:

(*a*) 30°, 80°. (*b*) 70°, 74°. (*c*) Each 60°.

6. Find the fourth angle of a quadrilateral if the other three angles are:

(*a*) 70°, 95°, 115°. (*b*) 130°, 126°, 77°. (*c*) Each 80°.

7. Find the fifth angle of a pentagon if the other four angles are:

(*a*) 80°, 90°, 100°, 110°. (*b*) 74°, 172°, 91°, 85°.
(*c*) Each 100°.

8. Given that the three angles of a triangle are equal, find the size of each.

9. If the four angles of a quadrilateral are equal, what is the size of each?

10. Given that the five angles of a pentagon are equal, find the size of each.

CONSTRUCTION OF TRIANGLES AND QUADRILATERALS

Exercises: V

Construct triangles ABC from the following data:

1. (*a*) AB = 4 in., BC = $3\frac{1}{2}$ in., CA = 3 in.

(*b*) AB = 6 cm., BC = 8 cm., CA = 7 cm.

2. (*a*) AB = 3 in., $\widehat{ABC}$ = 50°, BC = $3\frac{1}{2}$ in.

(*b*) CA = 6 cm., $\widehat{CAB}$ = 100°, AB = 7 cm.

3. (*a*) AB = 4 in., $\widehat{ABC}$ = 40°, $\widehat{BAC}$ = 35°.

(*b*) BC = 6 cm., $\widehat{ABC}$ = 70°, $\widehat{ACB}$ = 80°.

4. (*a*) AB = $3\frac{1}{2}$ in., $\widehat{ABC}$ = 70°, $\widehat{ACB}$ = 65°.

(*b*) BC = 5 cm., $\widehat{ABC}$ = 45°, $\widehat{BAC}$ = 55°.

5. (*a*) AB = 3 in., $\widehat{ABC}$ = 90°, CA = $4\frac{1}{2}$ in.

(*b*) CA = 6·5 cm., $\widehat{ACB}$ = 90°, AB = 8 cm.

6. (*a*) AC = 3 in., $\widehat{ACB}$ = 30°, AB = 2 in.

(*b*) BC = 4 cm., $\widehat{ABC}$ = 40°, CA = 6 cm.

7. (*a*) $\widehat{ABC} = 40°$, $\widehat{ACB} = 80°$, $\widehat{CAB} = 60°$.
(*b*) $\widehat{ACB} = 70°$, $\widehat{ABC} = 75°$, $\widehat{BAC} = 35°$.

8. (*a*) BC = 4·5 cm., $\widehat{ABC} = 90°$, AC = 8 cm.
(*b*) AC = 3 in., CB = 2·5 in., AB = 3·5 in.
(*c*) $\widehat{ABC} = 50°$, $\widehat{ACB} = 75°$, $\widehat{CAB} = 55°$.
(*d*) AB = 6·5 cm., $\widehat{ABC} = 80°$, $\widehat{ACB} = 70°$.
(*e*) AB = 5 cm., $\widehat{ABC} = 40°$, AC = 4 cm.
(*f*) BC = 4·5 in., $\widehat{ACB} = 120°$, CA = 2 in.
(*g*) CA = 3 in., $\widehat{CAB} = 55°$, BC = 3·5 in.

Draw the quadrilateral ABCD from the following data:

9. AB = 4 cm., $\widehat{ABC} = 80°$, BC = 3 cm., AD = 5 cm., CD = 4 cm.

10. AB = 2 in., BC = $1\frac{1}{2}$ in., AC = 2 in., AD = $2\frac{1}{2}$ in., CD = 1 in.

11. AB = 2 in., $\widehat{DAB} = 100°$, $\widehat{ABC} = 85°$, AD = 1 in., BC = $1\frac{1}{2}$ in.

12. AB = 4 cm., AD = 5 cm., BD = 5 cm., AC = 6 cm., BC = 4·5 cm.

13. AB = 2·5 in., $\widehat{DAB} = 80°$, $\widehat{ABC} = 85°$, AC = 3 in., $\widehat{ADC} = 100°$.

14. If A, B, and C are three posts such that AB = 400 yds., AC = 320 yds., and CB = 290 yds., what is the angle between AC and CB?

15. The point B is 700 ft. due east of the point A. A man sets out from A and walks in a direction N. 47° E. until he reaches a point C such that $\widehat{ACB} = 62°$. How far is C from B?

16. The straight roads AB and BC are at right angles, and BC is 400 yds. long. If A is 600 yds. from C, how far is A from B?

17. The point A is 3 miles from and N. 40° W. of B. A man sets out from A and walks in a direction S. 55° E. Find his bearing from B when he *first* reaches a point 1 mile from B.

18. If the man mentioned in the preceding exercise continues to walk in the same direction at 4 miles per hour, for how long is he less than 1 mile from A?

19. Two ships set out from the same port at the same time. The first sails at 10 m.p.h. due west. The second sails in a direction N. 63° W., and after 30 min. is 3 miles from the first

ship. Show from a diagram that the second ship must have one or other of two speeds. Find what these speeds are.

20. The points A, B, C, and D are such that AB = 400 yds., BC = 500 yds., AC = 520 yds., AD = 410 yds., and CD = 340 yds. Find the length of BD. How many solutions are there to this problem?

21. The points A, B, C, and D are such that AB = 40 ft., AD = 50 ft., $\widehat{DAB} = 120°$, $\widehat{CDA} = 80°$, and $\widehat{BCD} = 50°$. Find the length of BC.

22. The points A, B, and C lie, in this order, on a straight road. The point D, not on the road, is such that $\widehat{ADC} = 90°$ and $\widehat{BDC} = 60°$. If $\widehat{DAB} = 50°$ and AB = 400 yds., what is the length of BC?

23. A point B is 6 miles due east of C, and D is 8 miles from B and north of C. A ship sails from D in a direction N. 63° E. and reaches a point A such that $\widehat{DAB} = 80°$. What is the distance from B to A?

CONSTRUCTION OF SOLID FIGURES

EXERCISES: VI

1. Copy (*a*) Fig. 10, (*b*) Fig. 11, (*c*) Fig. 12, on stiff paper. Cut out each and fold at the broken lines. Fasten the surfaces together

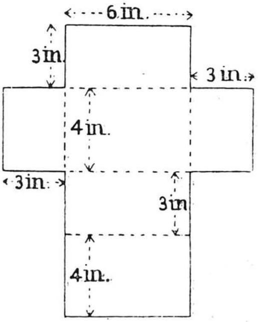


FIG. 10

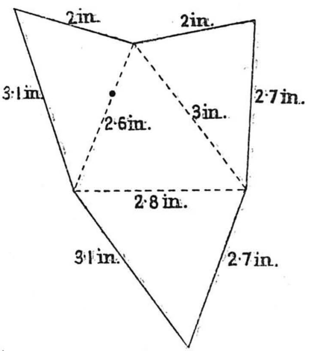


FIG. 11

with adhesive paper. The resulting solid figures are: (*a*) a *cuboid* (the ordinary box), (*b*) a *tetrahedron*, (*c*) a *triangular prism*.

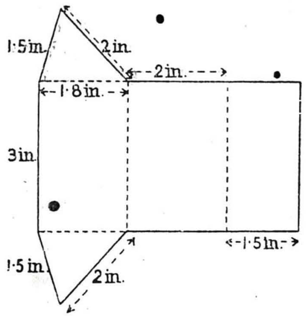


FIG. 12

2. Construct a box with equal edges which will hold one litre.

3. Draw the *net* of a *pyramid* on a square base given that the pyramid has equal edges. (*Cf.* Fig. 21.)

4. Copy Fig. 13 on stiff paper, and cut out the two shapes. Fold the *sector* ABC so that AB fits along AC. The circle will be

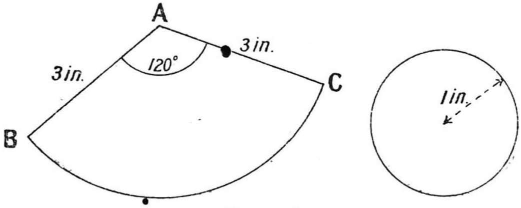


FIG. 13

found to fit on the base of the resulting solid, which is called a *cone*.

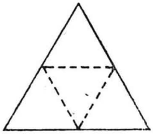


FIG. 14

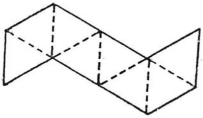


FIG. 15

5. Copy (*a*) Fig. 14, (*b*) Fig. 15, (*c*) Fig. 16, (*d*) Fig. 17, (*e*) Fig. 18,

making each edge 1 in. long. Cut out each shape and fold and fasten to make the five regular *polyhedra*. These are (*a*) the regular *tetrahedron*, (*b*) the regular *octahedron*, (*c*) the regular *icosahedron*, (*d*) the *cube*, (*e*) the regular *dodecahedron*.

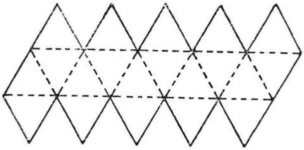


FIG. 16

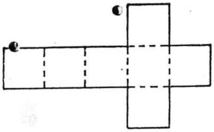


FIG. 17

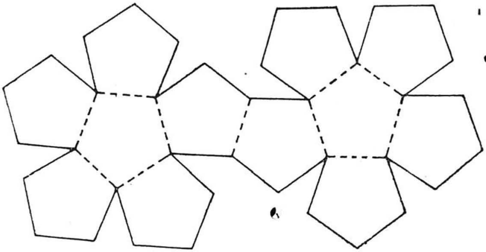


FIG. 18

For the remainder of this section models of the following solids should be available. If necessary they can be constructed in the manner indicated in the preceding examples.

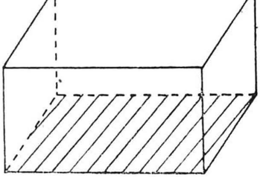


FIG. 19. CUBOID

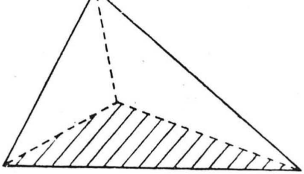


FIG. 20. TETRAHEDRON

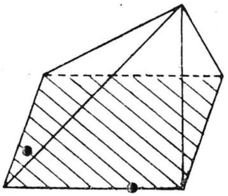


Fig. 21. Pyramid (with Square Base)

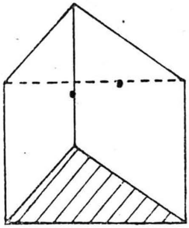


Fig. 22. Prism (with Triangular Base)

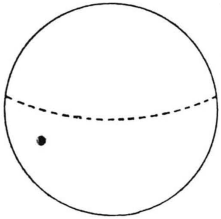


Fig. 23. Sphere

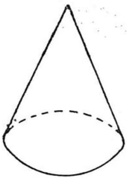


Fig. 24. Cone

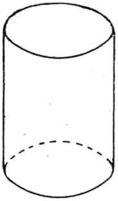


Fig. 25. Cylinder

Vertical, Horizontal

Craftsmen must have some means of testing whether surfaces are plane and whether they are horizontal or vertical, as required. To test if a piece of wood has a *plane surface* a joiner puts a straight edge of a ruler in different positions along the surface. If in all positions the straight-edge lies entirely along the surface he concludes that the surface is a plane surface. To test if a surface is horizontal a spirit-level is used. To secure a vertical line the bricklayer uses a plumb-line, which consists simply of a weight hanging freely on a length of string.

Exercises: VII

1. How many *faces* has (*a*) a cuboid, (*b*) a tetrahedron, (*c*) a square pyramid, (*d*) a triangular prism?

2. How many *edges* has each of the solids mentioned in the preceding question?

3. How many *vertices* has each of the solids mentioned in Exercise 1?

4. If the number of faces be denoted by F, the number of edges by E, and the number of vertices by V, what is the value of $F + V - E$ for each of the solids mentioned in Exercise 1?

5. How many plane surfaces bound (*a*) the cuboid, (*b*) the tetrahedron, (*c*) the square prism, (*d*) the cone, (*e*) the cylinder, (*f*) the sphere?

6. By applying the straight-edge to the cone show that it has one surface which is not plane. (This is called a *curved surface.*)

7. Determine whether the cylinder and the sphere have curved surfaces.

8. Satisfy yourself that the straight-edge can be placed in numerous positions all lying entirely along the curved surface of (*a*) a cone, (*b*) a cylinder. Can this be done if the cone is replaced by a sphere?

9. Name any four *horizontal* surfaces in the room.

10. Name any four *vertical* lines in the room.

11. Are all horizontal lines in the *same direction*?

12. Are all vertical lines in the same direction?

13. What is the size of the angle between two intersecting lines one of which is vertical and the other horizontal?

14. In the cuboid shown in Fig. 26 (*a*) are AB and EF *parallel—i.e.*, have they the same direction? (*b*) Are EF and DC parallel?

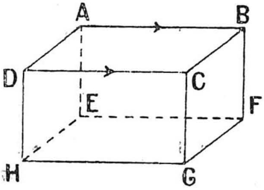


FIG. 26

15. Name five pairs of parallel edges in the above cuboid.

16. Name three pairs of parallel lines in the room.

SCALE DRAWING: THREE-DIMENSIONAL PROBLEMS

EXERCISES: VIII

It is recommended that graph paper be used for the following exercises.

1. A table 3 ft. wide and 6 ft. long is 2 ft. 6 in. high. Find the distance from the top of one leg to the bottom of the opposite leg.

2. Find the distance from the middle point of a long edge of the table given in the preceding exercise to the middle point of a leg on the opposite side.

3. In the box shown in Fig. 27 AB = 5 in., AD = 4in., and AE = 3 in.

(*a*) Draw DCGH and measure HC.

(*b*) Draw HEBC, and measure HB.

(*c*) Measure $\widehat{BHE}$.

4. Find the length of the shortest string which, starting at A, passes over the faces ABCD, DCGH, HGFE, and EFBA of the box in Fig. 27, and ends at B. (*Hint.* Draw the box opened out as in Fig. 17, p. 42.)

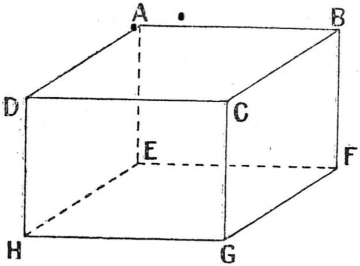


FIG. 27

5. Draw △ HGB of the box in Fig. 27 and measure $\widehat{BHG}$.

6. Given that the *slant height* of a cone is 6 in. and the *height* is 5 in. (Fig. 28), find the radius of the base.

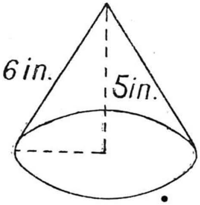


FIG. 28

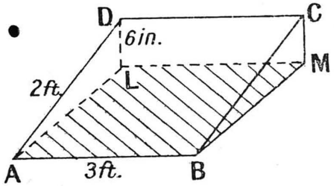


FIG. 29

7. Fig. 29 represents a sloping desk. Find (*a*) $\widehat{DAL}$, (*b*) $\widehat{CAB}$, (*c*) $\widehat{CAM}$.

8. A man standing due west of a tower finds its angle of elevation to be 27°. He walks 300 yds. north-east and then finds the tower is south-east of him. Find the height of the tower.

9. A man due south of a tower finds its angle of elevation to be 33°. After going 200 ft. due east he finds the tower is N. 42° W. of him. Find the height of the tower.

10. From an aeroplane 4000 ft. up two landmarks at sea-level

are seen. The first is due east, the second due south, and their angles of depression are 30° and 50°. Find the distance between them.

11. From the top of a tower 60 ft. high the angles of depression of two points on the ground, which is level, are 43° and 51°. The first point is south of the tower and the second is S. 48° E. of the tower. Find the distance between the points.

12. A is the foot of a wireless pole, and B is a point on the pole 15 ft. from A. BC and BD are supports each 20 ft. long, C and D being points on the level ground such that $\widehat{CAD} = 120°$. Find the length of the line joining C and D.

13. The feet of two goal-posts, AB and CD, are A and C, AC = 18 ft. and AB = 7 ft., and A is due west of C. Draw a plan of the shadow of the posts when the sun is in the south-east and its elevation is 36°.

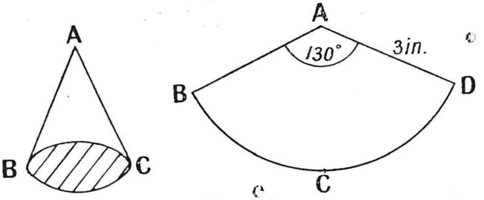


FIG. 30

14. ABCD represents the opened-out curved surface of the cone shown in Fig. 30. A fly crawls from B to C over the curved surface. Find the length of its shortest path.

15. ABCD is a tetrahedron with AB = 3·5 in., AC = 2·8 in., CB = 3 in., AD = 2 in., BD = 4 in., and CD = 2 in. By drawing the surfaces of the tetrahedron opened out as in Fig. 14 (p. 41) find the length of the shortest path along the surface from D to the mid-point of BC.

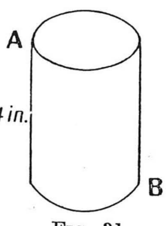


FIG. 31

16. If the length of the circumference of the cylinder in Fig. 31 is 6 in., what is the length of the shortest path from A to B along the surface of the cylinder?

17. Referring to the desk shown in Fig. 29 (p. 45), find the length of the shortest path from A to M: (*a*) along the surface ABML; (*b*) over the surfaces ABCD and DCML.

CONSTRUCTIONS

Exercises: IX

1. Draw an angle ABC of 72°. With centre B and any radius draw a circle to cut BA and BC at L and M. With centres at L and M and equal radii draw two circles to cut at K. Join BK. Verify, with a protractor, *that* BK *bisects* $\widehat{ABC}$.

2. Draw an angle of 64° and bisect it, using ruler and compasses. Verify by measurement.

3. Draw an angle of 130° and divide it into four equal parts, using ruler and compasses.

4. Draw a straight line AB. With centre A and any radius, draw a circle to cut AB at K. With centre K and the same radius draw a circle to cut the first circle at J. Join AJ. Verify, with a protractor, *that* $\widehat{JAB}$ *equals* 60°.

5. Use the construction given in the preceding exercise to make an angle of 120° with the aid of ruler and compasses.

6. Use the constructions given in Exercises 4 and 1 to make an angle of 30°.

7. Use the constructions given in Exercises 4 and 1 to make an angle of 90°.

8. Draw a straight line AB, 3 in. long. With centres A and B and equal radii draw two circles to cut at L and M. Join LM. Verify by measurement (*a*) that LM bisects AB, (*b*) that LM is perpendicular to AB. LM is called the *right bisector* of AB.

9. Draw a straight line 4 in. long. Use the construction given in Exercise 8 (*a*) to bisect it, (*b*) to divide it into four equal parts.

10. Draw a straight line AB and take any point C on it. With centre C and any radius draw a circle to cut AB at L and M. With centres L and M and a larger radius draw two equal circles to cut at P. Join PC. Verify *that* PC *is perpendicular to* AB.

11. Draw a straight line AB 5 in. long and take a point C on it 2 in. from A. Use the construction given in the preceding exercise to draw a line through C at right angles to AB.

12. Draw any straight line AB. Take any point C not on AB, and with centre C draw a circle to cut AB at P and Q. With centres P and Q and a new radius draw two equal circles to cut at L. Join CL. Verify *that* CL *is perpendicular to* AB.

13. Draw a △ ABC with sides 3 in., $2\frac{1}{2}$ in., and 2 in. long. Use the construction given in the preceding exercise to draw perpendiculars from A to BC, from B to CA, and from C to AB. Verify that these three perpendiculars meet in a point.

14. Draw a straight line AB. Take a point P outside AB. Draw the following three circles with equal radii:

(*a*) With centre P draw circle (i) to cut AB at K.
(*b*) With centre K draw circle (ii) to cut AB at J.
(*c*) With centre J draw circle (iii) to cut circle (i) at K and L.

Join PL. Verify, by means of ruler and set-square, *that* PL *is parallel to* AB.

15. Draw two straight lines AB and AC. Use the construction given in the preceding exercise to draw a line CX parallel to AB and a line BX parallel to AC. Verify by measurement that CX = AB and BX = AC.

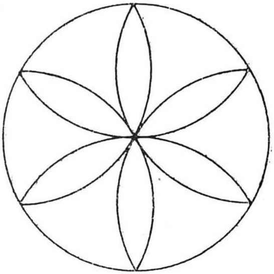


FIG. 32

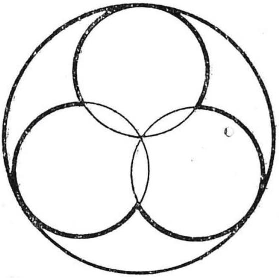


FIG. 33

16. Copy Fig. 32, using a radius of 2 in. for each circle.

17. Copy Fig. 33, using a radius of 2 in. for the large circle.

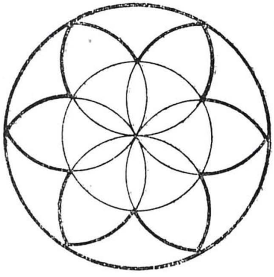


FIG. 34

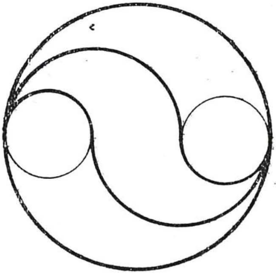


FIG. 35

18. Copy Fig. 34, using a radius of 1 in. for the inner circle.

19. Copy Fig. 35, using a radius of $1\frac{1}{2}$ in. for the large circle.

LOCUS

If O (Fig. 36) represents a stake to which a donkey is tied by a rope 30 yds. long, then it is clear that if the donkey moves, keeping the rope tight, it will describe a circle, centre O and radius 30 yds.

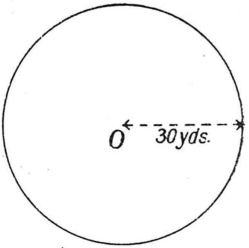


Fig. 36

This circle is said to be the *locus* of a moving point which is always 30 yds. from the fixed point O.

When we speak of the locus of the donkey being a circle we mean two things:

(*a*) If the donkey is anywhere on the circle it is 30 yds. from O.

(*b*) If the donkey is 30 yds. from O it is on the circle.

(The plural of 'locus' is 'loci.')

Exercises: X

Graph paper may be used for the following exercises.

1. Mark two points A and B to represent two trees 300 yds. apart. Take 1 in. to represent 100 yds. Find the position of a man, say P, who is 225 yds. from A and 225 yds. from B. Find a position Q which is 290 yds. from A and from B. Find a position R which is 330 yds. from A and from B. Find a position S which is 400 yds. from A and from B. Verify that P, Q, R, and S lie on a straight line. Produce this line both ways. Take any point K on the line or the produced line. Measure AK and BK. Are they equal? Can you find a point X such that AX equals BX and X is not on the line PS? The line PS is the locus of a man who moves keeping *equidistant* from the two trees.

2. Draw a straight line AB and produce it both ways. Find six positions of a point which is 1 in. from this line. Draw the locus of such a point.

3. What is the locus of the hub of the wheel of a bicycle which is travelling along a straight, level road?

4. An enclosed rectangular field ABCD has AB = 80 yds. and AD = 40 yds. Draw a plan of the field and show the locus

of a donkey which is tethered to A by means of a taut rope 50 yds. long.

5. Draw a plan of a table 8 ft. long and 5 ft. wide. Draw the locus of a fly which crawls on the table keeping 1 ft. from an edge.

6. Draw a plan of a rug 5 ft. long and 3 ft. wide. Draw the locus of a fly which crawls on the floor around the rug keeping 1 ft. from the nearest edge.

7. A piece of cotton 4 ft. long is stretched on the floor. Draw the locus of a fly which moves on the floor keeping 1 ft. from the cotton.

8. If the same piece of cotton is suspended in space, what is the locus of a fly which can move anywhere in space keeping 1 ft. from the cotton?

9. A ladder 20 ft. long slips outward with its ends in contact with a vertical wall and a horizontal floor. Draw the locus of the mid-point of the falling ladder.

10. Draw a straight line AB of any length. Take a fixed point C 4 in. from AB. Given that a point P moves along the line AB, draw the locus of the mid-point of CP.

11. Draw two lines AB and CD at right angles to represent two roads. Draw the locus of a man who moves keeping 10 yds. from one or other of the roads.

12. Draw two lines AB and AC at right angles. AB represents a vertical wall and AC represents a straight line on the horizontal ground. What is the locus of the centre of the front wheel of a bicycle which is being ridden along AC towards AB? What is the locus of the centre of the back wheel?

13. Place two pins at points A and B 9 cm. apart. Connect A and B by a string 12 cm. long. Keeping the string taut, draw the locus obtained by allowing a pencil-point to move in contact with the string. (The curve you get is called an *ellipse*.)

14. Verify that an exactly similar ellipse is obtained as follows. Draw a line AB and take a point S $3\frac{1}{2}$ cm. from AB. Plot a number of positions of a point P such that $SP = \frac{3}{4}PM$, where PM is perpendicular to AB. Hence draw the locus of the point P.

15. A lighthouse is 4 miles from a straight stretch of shore. Draw a plan showing the locus of a boat which moves keeping the same distance from the shore as from the lighthouse. (The curve showing this locus is called a *parabola*.)

16. Given that the distance of the boat from the lighthouse in the preceding exercise has to be twice its distance from the shore, draw a plan to show the new locus of the boat. (The curve obtained is called a *hyperbola*.)

17. Draw a circle of radius 1 in., and take a fixed point X on the circumference. Given that P is a point which moves on the circumference of the circle, draw the locus of the mid-point of XP.

18. Given that the line XP of the preceding exercise (produced if necessary) has a point Q on it, such that PQ = 1 in., draw the locus of Q. (The curve obtained is known as a *limaçon.*)

19. Draw the locus of a pencil-point attached to the end of a piece of cotton which is being unwound from a fixed reel. (The curve obtained is called a *spiral.*)

20. Copy Fig. 37, and draw the locus of a point which moves keeping $2\frac{1}{2}$ cm. from the nearest boundary of the cross.

21. Draw on your copy of Fig. 37 the locus of a point which moves keeping 1 cm. from the nearest boundary of the cross.

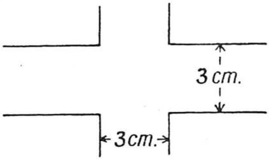


Fig. 37

22. Draw a straight line AB 2 in. long. Make △ APB with $\widehat{PAB} = 40°$ and $\widehat{PBA} = 50°$. How big is $\widehat{APB}$? Find another point Q so that $\widehat{AQB} = 90°$. Draw the locus of a point R which moves so that $\widehat{ARB} = 90°$.

23. A marble rolls inside a spherical bowl. What is the locus of its centre?

24. Draw a rough sketch showing the locus of a cricket-ball which is 'skied' by a batsman.

25. A set-square ABC is held in a vertical plane with AB resting on a horizontal table. The set-square is turned about B until BC rests on the table, and is then turned about C until CA rests on the table. During both these operations the set-square is kept in the original vertical plane. Sketch the locus of A.

26. Sketch the locus of the point A of a page ABCD which starts with AB on a horizontal table and is turned about B, C, and D in turn until DA rests on the table, the page remaining in the same vertical plane throughout.

PART I

FUNDAMENTAL IDEAS

SOLIDS, SURFACES, LINES, POINTS, AND ANGLES

Summary of Facts

The cuboid (shown in Fig. 38) is one of many solid figures. It is bounded by six plane surfaces, of which ABCD is one.

The two plane surfaces ABCD and DCGH meet in the straight line CD, which we shall call CD, or the line CD.

The two lines DC and CG meet in a point—namely, C.

The two lines DC and AB are in the same plane, but do not meet however far they are produced; they are said to be **parallel** lines.

The two lines HD and CB do not meet, but as they are not in the same plane they are not parallel lines.

The arrows in the figure are used to denote that AB is parallel to DC, which we write for short as AB || DC.

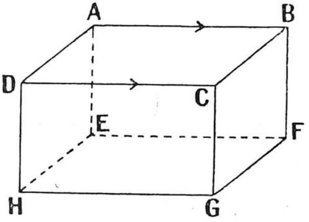


Fig. 38

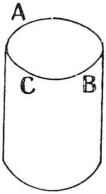


Fig. 39

The solid illustrated in Fig. 39 is a cylinder. It is bounded by three surfaces, two of which are plane—namely, the ends—and one curved. The curved surface meets one of the plane surfaces in the curved line ABC.

Fundamental Facts concerning Straight Lines and Planes

The pupil should satisfy himself that he agrees with the following assumptions, since they will be used later in formal work:

***Assumption* 1. Through two given points one, and only one, straight line can be drawn.**

***Assumption* 2. The shortest path between two given points is the straight line joining them.**

***Assumption* 3. A surface is plane if the straight line joining any two points on the surface lies wholly on the surface.**

Angles at a Point

If any line revolves from an original position OX to a new position OY, then it has turned through an angle XOY—*i.e.*, an angle is an amount of turning. This amount of turning is not altered if OX and OY are of different lengths from those shown in Fig. 40.

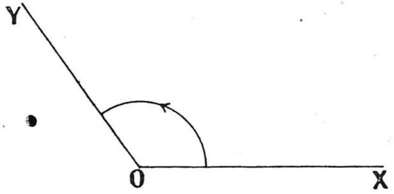


Fig. 40

If OY turns completely round O once, OX being its original and final position (Fig. 41), we say that OY has turned through one revolution. One revolution is divided into 360 equal angles, each of which is called a degree, and the degree is divided into sixty equal angles, called minutes. For short, 360 degrees is written 360°, and 60 minutes is written 60′. We can therefore write

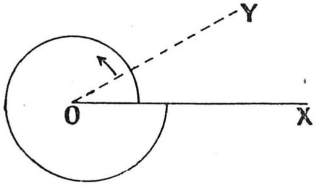


Fig. 41

$$1 \text{ revolution} = 360°$$

and $$1° = 60'.$$

Right Angle. One right angle is a quarter of a revolution; *i.e.*, 1 right angle = 90°.

PERPENDICULAR, HORIZONTAL, AND VERTICAL LINES

Two lines in one plane meeting at right angles are said to be mutually *perpendicular*.

A motionless string having a weight attached to one end and suspended freely from the other is said to define a *vertical* line. The line, if continued, will pass through the centre of the earth, and is called a plumb-line. The string itself is not a line, but a very thin cylinder. The finer the string the more nearly does it approximate to a line.

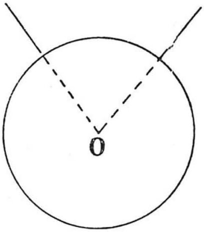


FIG. 42

Plumb-lines in Manchester and New York are both vertical lines. They are clearly not parallel, since they can be produced to meet at the centre of the earth. (See Fig. 42.)

Vertical lines in the same room are very nearly parallel, and only a small error is made by assuming them to be parallel.

Any plane having each line in it perpendicular to a given vertical line is a *horizontal* plane.

If OY turns from its original position OX to the position OZ, such that XOZ is a straight line, it is clear that the angle described is half a revolution—*i.e.*, 180°. (See Fig. 43.) This angle is sometimes called a straight angle. In this book such an angle will be referred to as **half a revolution.**

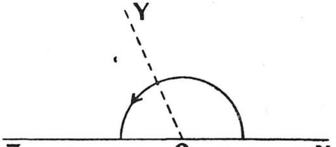


FIG. 43

The point O is the **vertex** of the angle XOY; OX and OY are the **arms** of the angle. Angle XOY will be written $\widehat{XOY}$.

The symbols > and <

$a > b$ means 'a is greater than b.'
$a < b$ means 'a is less than b.'

Acute, Obtuse, and Reflex Angles

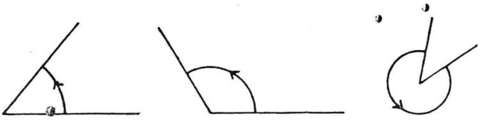


Fig. 44

An *acute* angle is an angle $< 90°$.
An *obtuse* angle is an angle $> 90°$ and $< 180°$.
A *reflex* angle is an angle $> 180°$ and $< 360°$.

Adjacent Angles

If two angles have a common arm and a common vertex and are on opposite sides of the common arm they are called *adjacent* angles.

Thus, x and y in Fig. 45 are adjacent angles. $\widehat{XOZ}$ in Fig. 45 is obtained by a line revolving from the position OX to OY and then from OY to OZ, and we say that

$$\widehat{XOZ} = \widehat{XOY} + \widehat{YOZ}.$$

Hence $\widehat{XOY} = \widehat{XOZ} - \widehat{YOZ}.$

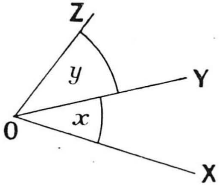


Fig. 45

Note. In Fig. 45 there are two angles XOZ. When we speak of $\widehat{XOZ}$ we mean the angle which is less than 180°; if we wish to refer to the angle which is greater than 180° we call it the reflex $\widehat{XOZ}$.

Complementary and Supplementary Angles

Two angles whose sum is 90° are called *complementary* angles, and each angle is said to be the *complement* of the other.

Two angles whose sum is 180° are called *supplementary* angles, and each angle is said to be the *supplement* of the other.

VERTICALLY OPPOSITE ANGLES

If in Fig. 46 AB and CD are straight lines cutting at O,

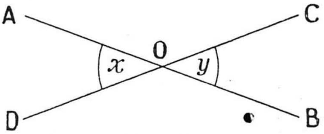


FIG. 46

then x and y are called *vertically opposite* angles. Similarly, $\widehat{AOC}$ and $\widehat{BOD}$ are vertically opposite angles.

BISECTION

To *bisect* means to divide into two equal parts. Thus, if AB and XY in Fig. 47 intersect (*i.e.*, cut) at C so that the length of AC equals the length of CB (*i.e.*, AC = CB), then XY bisects AB.

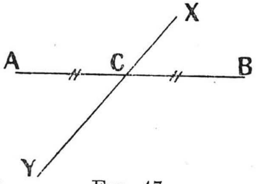


FIG. 47

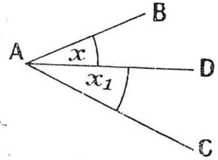


FIG. 48

Again, if in Fig. 48 x equals x_1, then AD is said to bisect $\widehat{BAC}$.

ANGLE ASSUMPTIONS

The pupil should satisfy himself that he agrees with the following assumptions, as they will be used later in formal work:

***Assumption* I. If a straight line be drawn from a point on another straight line the sum of the angles so formed is two right angles.**

Thus, if in Fig. 49 AB is

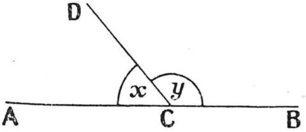


FIG. 49

a straight line with C a point on it from which the line CD is drawn, then $x + y = 2$ right angles.

***Assumption* 2. If from a point on a straight line two straight lines be drawn on opposite sides of the first line to make the sum of the two adjacent angles equal to two right angles, then these two straight lines are in one and the same straight line.**

That is, if in Fig. 49 $x + y$ equals two right angles, then ACB is a straight line.

***Assumption* 3. If two straight lines intersect the vertically opposite angles are equal in pairs.**

Thus, in Fig. 50,

(i) $x = y$ and (ii) $a = b$.

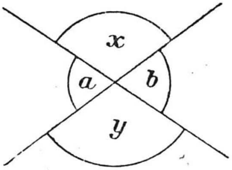


FIG. 50

EXERCISES: XI

1. In Fig. 51 ACB is a straight line. Find x when:

(a) $y = 40°$.
(b) $y = 80°$.
(c) $y = 118°$.
(d) $y = \frac{1}{3}$ rt. angle.
(e) $y = 1\frac{1}{2}$ rt. angles.

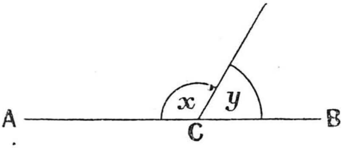


FIG. 51

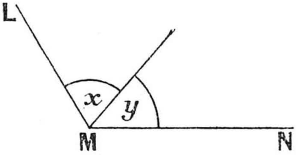


FIG. 52

2. In which of the following cases can we say that LMN in Fig. 52 is a straight line?

(a) $x = 40°$, $y = 130°$.
(b) $x = 40°$, $y = 140°$.
(c) $x = 80°$, $y = 100°$.
(d) $x = \frac{1}{2}$ rt. angle, $y = 2$ rt. angles.
(e) $x = 1\frac{1}{4}$ rt. angles, $y = \frac{3}{4}$ rt. angle.
(f) $x = 30°$, $y = 5x$.
(g) x and y are supplementary.
(h) x and y are complementary.

3. In Fig. 53 ACB is a straight line. Calculate the value of x in degrees.

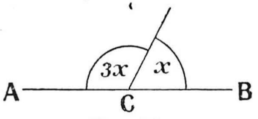


FIG. 53

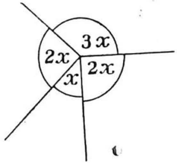


FIG. 54

4. Calculate the value in degrees of x in Fig. 54.

5. Calculate the value of y in degrees, given that in Fig. 55 (*a*) $x = 40°$, (*b*) $x = 50°$.

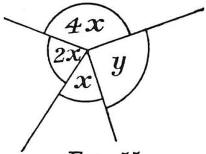


FIG. 55

6. Find the angle through which the hour hand of a clock turns from:

(*a*) 12 o'clock to 2 o'clock.
(*b*) 1 o'clock to 10 o'clock.
(*c*) 1 o'clock to half-past 3.

7. Find the angle between lines in the following directions:

(*a*) N.E. and N.W.
(*b*) E.N.E. and W.N.W.
(*c*) W.S.W. and N.N.E.
(*d*) N. 34° E. and S. 48° E.
(*e*) S. 22° W. and N. 18° E.

8. The line OK is drawn within $\widehat{MON}$, OX bisects $\widehat{NOK}$, and OY bisects $\widehat{MOK}$. What is the value of $\widehat{XOY}$ in degrees (*a*) when $\widehat{MOK} = 40°$ and $\widehat{NOK} = 110°$, (*b*) when $\widehat{NOM} = 156°$?

9. Given that AOB and COD in Fig. 56 are straight lines and $\widehat{AOC}$ equals 90°, calculate the value of $\widehat{XOY}$ in degrees.

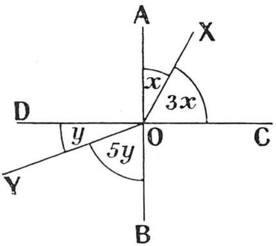


FIG. 56

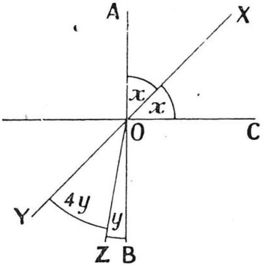


FIG. 57

10. Given that AOB and XOY in Fig. 57 are straight lines and $\widehat{AOC}$ equals 90°, calculate the value of $\widehat{XOZ}$ in degrees.

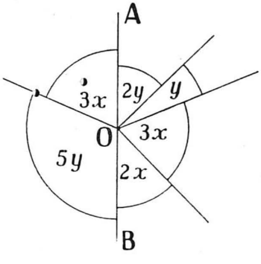


FIG. 58

11. The lines OA, OB, and OC are taken in this order so that $\widehat{AOB} = 90°$ and $\widehat{BOC} = 70°$. The line OX is taken inside $\widehat{AOB}$ so that $\widehat{AOX} = \frac{1}{2}\ \widehat{BOX}$, and the line OY divides the *reflex* $\widehat{AOC}$ so that $\widehat{AOY} = 3\ \widehat{YOC}$. Prove that XOY is a straight line.

12. Given that AOB in Fig. 58 is a straight line, calculate the values of x and y in degrees.

CORRESPONDING, ALTERNATE, AND INTERIOR ANGLES

Definitions. If in Fig. 59 AB and CD are two straight lines cut by another straight line LM, then:

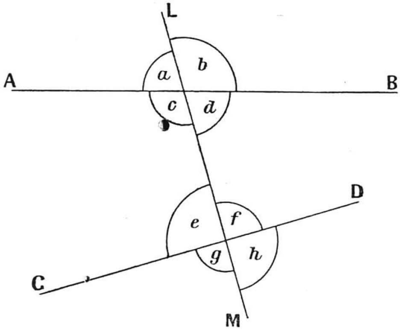


FIG. 59

I. a and e are said to be *corresponding angles.*
Similarly, c and g, b and f, d and h, are corresponding angles.

II. c and f are said to be *alternate angles.*
Similarly, d and e are alternate angles.

III. c and e are said to be *interior angles on the same side of the cutting line,* as are d and f.

A pair of interior angles on the same side of the cutting line will be referred to simply as *interior angles.*

FACTS ABOUT PARALLELS

If a set-square be moved along a ruler's edge from the position ABC to the position DEF (Fig. 60) we shall assume that AB and DE are straight lines in the same direction.

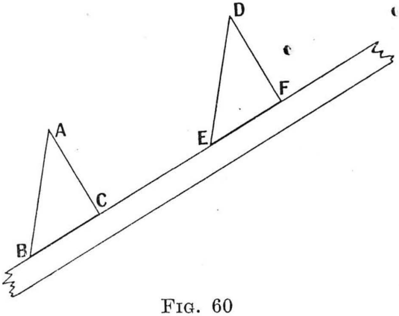


FIG. 60

Since $\widehat{ABC} = \widehat{DEF}$, this assumption is equivalent to the first of the following assumptions:

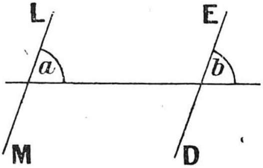


FIG. 61

Assumption 1. If two straight lines are cut by a third straight line so as to make a pair of corresponding angles equal, then the first two lines are parallel.

Thus, if in Fig. 61

$$a = b,$$

then LM || ED.

Assumption 2. If two parallel straight lines are cut by a third straight line, then any pair of corresponding angles are equal.

Thus, if in Fig. 61

LM || ED,

then $a = b$.

Assumption 3. If two straight lines are cut by a third straight line so as to make a pair of alternate angles equal, then the first two lines are parallel.

Thus, if in Fig. 62

$$a = y,$$

then LM || ED.

Assumption 4. If two parallel straight lines are cut by a third straight line, then any pair of alternate angles are equal.

Thus, if in Fig. 62

LM || ED,

then $a = y$.

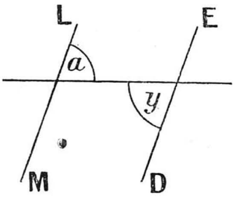


FIG. 62

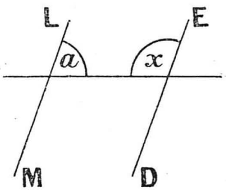


FIG. 63

Assumption 5. If two straight lines are cut by a third straight line so as to make a pair of interior angles supplementary, then the first two lines are parallel.

Thus, if in Fig. 63,

$$a + x = 180°,$$

then LM || ED.

Assumption 6. If two parallel straight lines are cut by a third straight line, then any pair of interior angles are supplementary.

Thus, if in Fig. 63

LM || ED,

then $a + x = 180°$.

Assumption 7. If two straight lines are parallel to the same straight line they are parallel to each other.

Thus, in Fig. 64

if AB || EF

and CD || EF,

then AB || CD.

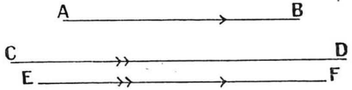


FIG. 64

Once again the pupil should satisfy himself that the preceding statements are true.

NOTE. By assuming the truth of Assumption 1 the pupil should be able to show that Assumptions 3 and 5 are true; by assuming the truth of Assumption 2 that Assumptions 4 and 6 are true; and by assuming the truth of Assumptions 1 and 2 that Assumption 7 is true.

EXAMPLE

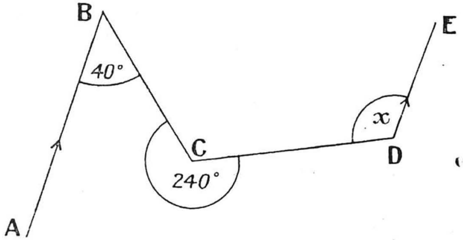


FIG. 65

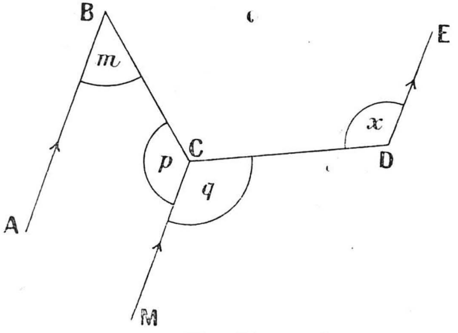


FIG. 66

Given: Fig. 65, in which AB is parallel to DE.
To find: x.
Work: Draw Fig. 66—*i.e.*, Fig. 65 with

$$CM \parallel BA \parallel ED.$$

Since $m = 40°$ (given)
and $p + m = 180°$ (interior angles; BA || CM),
$\therefore\ p = 140°$.
$\therefore\ q = 100°$ (since $p + q = 240°$).
Also $q = x$ (alternate angles; CM || ED).
$\therefore\ x = 100°$.

EXERCISES: XII

Section A

1. State, with reason, whether AB and CD in Fig. 67 are parallel in each of the following cases:

(*a*) $p = 70°$, $k = 69°$.
(*b*) $q = 120°$, $k = 64°$.
(*c*) $p = 100°$, $l = 80°$.
(*d*) $q = 79°$, $n = 101°$.
(*e*) $p = 120°$, $n = 60°$.

2. Say whether AB and CD in Fig. 67 are parallel when:

(*a*) $p = 130°$, $l = 50°$.
(*b*) $p = 120°$, $l = 70°$.
(*c*) $p = 90°$, $l = 90°$.

If AB and CD are not parallel in any one of these three cases, state on which side of EF the lines meet.

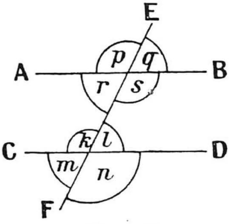


FIG. 67

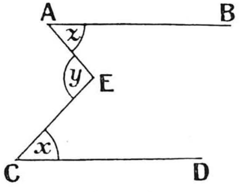


FIG. 68

3. State whether AB and CD in Fig. 68 are parallel when:

(*a*) $x = 40°$, $y = 90°$, $z = 50°$.
(*b*) $x = 25°$, $y = 70°$, $z = 40°$.
(*c*) $x = 90°$, $y = 80°$, $z = 12°$.

If AB and CD are not parallel state in which direction they meet.

4. State whether AB and CD in Fig. 69 are parallel when:

(*a*) $l = 130°$, $m = 40°$, $n = 100°$.
(*b*) $l = 120°$, $m = 60°$, $n = 120°$.
(*c*) $l = 110°$, $m = 50°$, $n = 210°$.

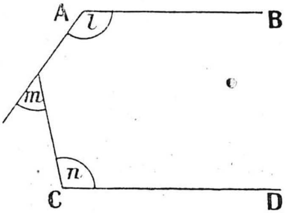


FIG. 69

5. If we do not give particular values to l, m, and n in Fig. 69, what equation must connect l, m, and n for the lines AB and CD to be parallel?

6. (*a*) Given in Fig. 70 that $y = 40°$, find the value of x.
(*b*) In Fig. 71, if $y = 120°$, what is the value of x?

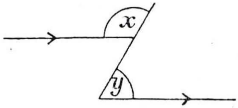


FIG. 70

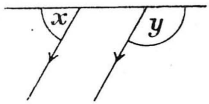


FIG. 71

(*c*) Find the value in degrees of x in Fig. 71 given that $y = 2x$.
(*d*) Find the value of x in Fig. 72 when $y = 110°$.

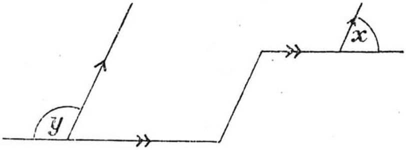


FIG. 72

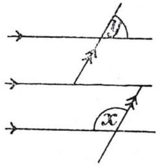


FIG. 73

(*e*) Given in Fig. 73 that $y = 70°$, find the value of x.
(*f*) In Fig. 73, if $x = 3y$ what is the value in degrees of y?

(*g*) Find the value of x in Fig. 74 when $y = 40°$.

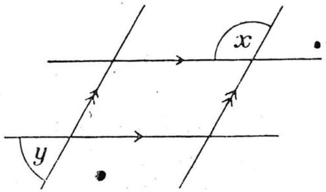


Fig. 74

7. For each of Figs. 70–74 find an equation connecting x and y.

8. Given that in Fig. 75 AB is parallel to CD, prove that $x = y$.

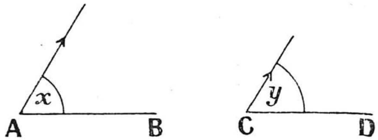


Fig. 75

9. State whether AB and CD in Fig. 75 are parallel when:

(*a*) $x = 40°$, $y = 50°$. (*c*) $x = 80°$, $y = 100°$.
(*b*) $x = 70°$, $y = 70°$.

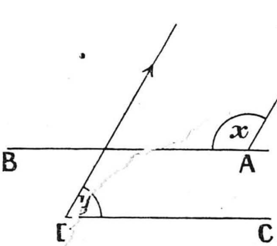


Fig. 76

10. In each of the following cases find whether AB and CD in Fig. 76 are parallel when:

(*a*) $x = 120°$, $y = 70°$. (*c*) $x = 100°$, $y = 95°$.
(*b*) $x = 90°$, $y = 90°$.

Section B

11. State whether AB and CD in Fig. 77 are parallel in each of the following cases:

(*a*) $p = 25°$, $q = 165°$.
(*b*) $p = 90°$, $q = 80°$.
(*c*) $p = 100°$, $q = 80°$.

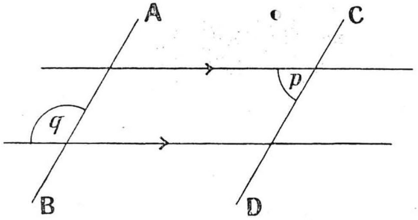


Fig. 77

12. What relation must hold between p and q in Fig. 77 for AB and CD to be parallel?

13. If in Fig. 78 $a = a_1 = 50°$, what are the values of x and y?

14. Given that in Fig. 78 $a = a_1$, find an equation connecting x and y.

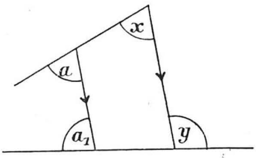


Fig. 78

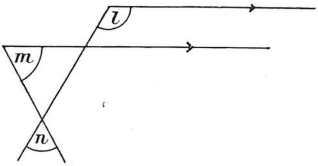


Fig. 79

15. Given that in Fig. 79

(*a*) $l = 120°$, $m = 60°$, find n.
(*b*) $m = 50°$, $n = 70°$, find l.
(*c*) $l = 140°$, $n = 60°$, find m.

Find an equation connecting l, m, and n in Fig. 79.

16. Given three straight lines, AB, BC, and CA with $\widehat{A} = 70°$, $\widehat{B} = 45°$, BC produced to D, and CE drawn parallel to BA, find the value in degrees of (*a*) $\widehat{ACE}$, (*b*) $\widehat{ECD}$, (*c*) $\widehat{ACD}$, (*d*) $\widehat{ACB}$.

17. If in Fig. 80 AB ∥ CD, $l = 120°$, and $3y = 4x$, what is the value of m?

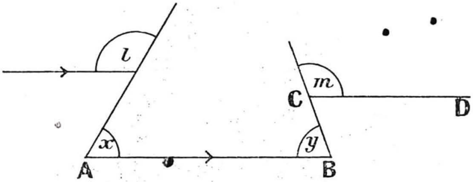


Fig. 80

18. If in Fig. 80 $3x = 2y$, $l = 120°$, and $m = 80°$, show that CD is not parallel to AB. In which direction do the two lines meet?

19. Given that in Fig. 81 AB is parallel to CD, find an equation connecting (*a*) x, y, and z; (*b*) y and k.

20. Using the information given in Fig. 82, find an equation connecting (*a*) m, k, and z; (*b*) k, x, and y; (*c*) k, x, and z.

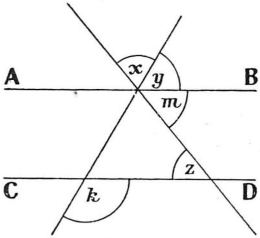


Fig. 81

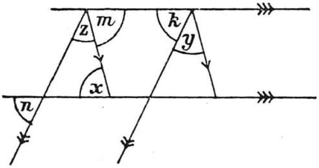


Fig. 82

21. State whether AB and CD in Fig. 81 are parallel when:

(*a*) $x = 40°$, $y = 70°$, $k = 100°$.
(*b*) $x = 40°$, $y = 70°$, $z = 60°$.
(*c*) $x = 40°$, $y = 70°$, $k = 110°$.
(*d*) $x = 50°$, $m = 60°$, $k = 110°$.

22. Prove that the following method of drawing through a given point P a straight line parallel to a given straight line AB is correct:

(i) Choose any point C on AB.
(ii) Join CP.
(iii) Measure $\widehat{ACP}$ with a protractor.
(iv) Make $\widehat{CPD} = \widehat{ACP}$, with D and B on the same side of AB.

Then PD is the required line parallel to AB.

DEFINITIONS

TRIANGLES

A figure bounded by three straight lines is called a *triangle.* For example, in Fig. 83 ABC is a triangle. A is called a *vertex* of the triangle. Similarly, B and C are vertices of the triangle. AB is called a *side* of the triangle; BC and CA are also sides of the triangle. A side of a triangle is sometimes referred to as the *base* of the triangle.

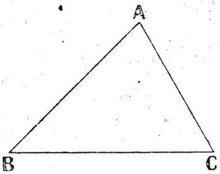


FIG. 83. ACUTE-ANGLED TRIANGLE

If in Fig. 83 AC be chosen as the base of the triangle the angles at A and C are referred to as *base angles,* and the angle at B as the *vertical angle* of the triangle.

When each angle of the triangle is acute, as in Fig. 83, the triangle is said to be an *acute-angled triangle.*

If one angle of a triangle be obtuse the triangle is said to be an *obtuse-angled triangle.*

In Fig. 84 the angle at A is obtuse.

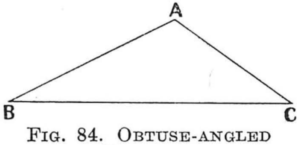


FIG. 84. OBTUSE-ANGLED TRIANGLE

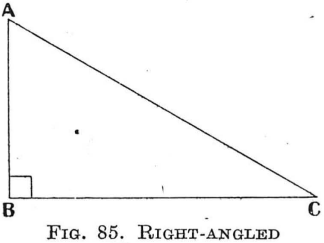


FIG. 85. RIGHT-ANGLED TRIANGLE

If one angle of a triangle be a right angle the triangle is said to be a *right-angled triangle.*

The side opposite the right angle is called the *hypotenuse* of the triangle.

In Fig. 85 the angle at B is a right angle. Therefore AC is the hypotenuse.

Quadrilateral. A figure bounded by four straight lines is called a *quadrilateral* (Fig. 86).*

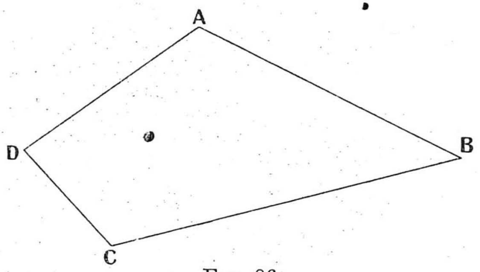


FIG. 86

The symbol △ is frequently used for 'triangle,' and 'quadrilateral' is sometimes abbreviated to 'quad[l].'

Diagonal. The line joining two opposite vertices of a quadrilateral is called a *diagonal* of the quadrilateral. Thus a quadrilateral has two diagonals.

In Fig. 87 AC is a diagonal of the quadrilateral ABCD.

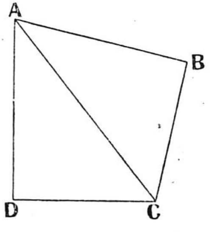


FIG. 87

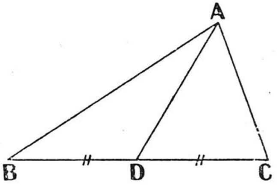


FIG. 88

Median. The line joining a vertex of a triangle to the middle point of the opposite side is called a *median* of the triangle. A triangle has three medians.

In Fig. 88 AD is a median of the △ ABC.

* The following particular quadrilaterals are defined on page 98: trapezium, parallelogram, rhombus, rectangle, and square.

Exterior Angles. If the sides of a $\triangle$ ABC are produced we have the figure shown in Fig. 89. At the vertex A the two

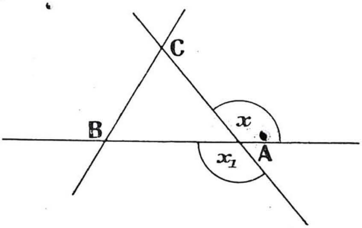


FIG. 89

angles x and x_1 are called *exterior* angles—*i.e.*, an exterior angle of a triangle is an angle between one side of the triangle and another side produced.

As x and x_1 are vertically opposite angles they are equal.

There are also two equal exterior angles at B and two equal exterior angles at C.

Similarly, in the five-sided figure shown in Fig. 90, a, b, c, d, and e are interior angles, and v, w, x, y, and z are exterior angles.

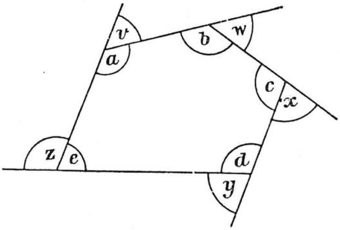


FIG. 90

PART II
GEOMETRICAL THEOREMS

So far we have been dealing with fundamental facts in the study of geometry, and these we accept without proof. We now come to a new group of geometrical facts, the theorems, each of which requires systematic proof.

THEOREM 1

If one side of a triangle be produced the exterior angle so formed is equal to the sum of the two opposite interior angles of the triangle. The sum of the three angles of the triangle is two right angles.

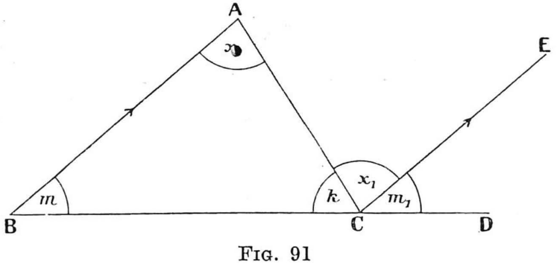


FIG. 91

Given a △ ABC with BC produced to D.

Prove that (i) $\widehat{ACD} = x + m$,

(ii) $x + m + k = 180°$.

Construction. Draw CE || BA.

Proof. (i) We know that

$x = x_1$ (alt. angles; BA || CE),

and $m = m_1$ (corr. angles; BA || CE).

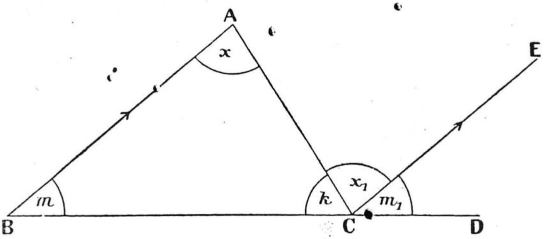


FIG. 91

Therefore, by addition,

$$x + m = x_1 + m_1,$$

i.e., $x + m = \widehat{ACD}$. *Q.E.D.*

(ii) We know that

$$x = x_1 \quad \text{(alt. angles; BA} \parallel \text{CE)},$$
$$m = m_1 \quad \text{(corr. angles; BA} \parallel \text{CE)},$$
$$\text{and} \quad k = k.$$
$$\therefore \quad x + m + k = x_1 + m_1 + k.$$

Also $x_1 + m_1 + k = 180°$ (half a revolution).

$\therefore \quad x + m + k = 180°$. *Q.E.D.*

Corollary. Each exterior angle of a triangle is greater than either of the two interior and opposite angles.

Proof. In Fig. 91

$$\widehat{ACD} = x + m.$$
$$\therefore \quad \widehat{ACD} > x,$$
$$\text{and} \quad \widehat{ACD} > m.$$

NOTE. In order to use single letters for angles without confusing the figure, we complete the construction before lettering the angles. In the above theorem we draw the parallel CE before lettering the angles, and do not denote each of the three angles ACE, ECD, and ACD by a single letter.

ENUNCIATIONS, GENERAL AND PARTICULAR

The statement of a theorem in general terms is called the *general enunciation* of the theorem. The statement of the theorem in terms of a named figure is called the *particular enunciation* of the theorem. Each enunciation consists of two parts: (i) a statement of what is given, sometimes called the *datum* or *hypothesis*; (ii) a statement of the fact or facts that may be deduced from the given facts. This statement is called the *conclusion* of the theorem.

The letters *Q.E.D.*, which are put at the end of a theorem, stand for the Latin *quod erat demonstrandum* and mean 'which was to be proved.'

A *corollary* to a theorem is a further fact which may be deduced from the theorem.

THEOREM 1A

The sum of the interior angles of a quadrilateral is four right angles.

The proof of this theorem is left to the pupil. (*Hint.* Form two triangles by drawing a diagonal.)

Polygons. Any plane closed figure whose sides are straight lines is a *polygon*.

Figs. 92 and 93 show a six-sided *convex* polygon and a six-sided *re-entrant* polygon respectively. The distinction is that

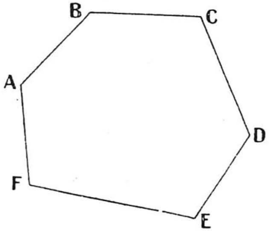


FIG. 92

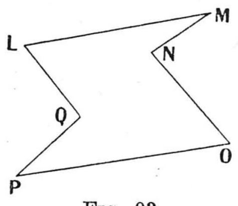


FIG. 93

Fig. 92 has none of its interior angles greater than 180°, while Fig. 93 has at least one of its interior angles greater than 180°.

A polygon is said to be *regular* if all its sides are equal and also all its angles are equal. Thus, for Fig. 92 to be a regular polygon we must have

$$AB = BC = CD = DE = EF = FA$$

and also $\widehat{A} = \widehat{B} = \widehat{C} = \widehat{D} = \widehat{E} = \widehat{F}.$

A	3-sided polygon is called	a	triangle.
,,	4 ,, ,, ,, ,,	,,	quadrilateral.
,,	5 ,, ,, ,, ,,	,,	pentagon.
,,	6 ,, ,, ,, ,,	,,	hexagon.
,,	7 ,, ,, ,, ,,	,,	heptagon.
An	8 ,, ,, ,, ,,		an octagon.
A	9 ,, ,, ,, ,,		a nonagon.
,,	10 ,, ,, ,, ,,	,,	decagon.

Exercises: XIII

Section A

1. Given that in Fig. 94

(*a*) $x = 70°$ and $q = 10°$, find the values of p and r.
(*b*) $p = 20°$ and $q = 40°$, find the values of r and x.
(*c*) $r = 100°$ and $p = 10°$, find the value of q.
(*d*) $p = 90°$ and $x = 130°$, find the values of q and r.

2. In the data given below the letters represent angles as marked in Fig. 94. In which of the cases given can triangles be drawn?

(*a*) $q = 20°, p = 40°, r = 110°$.
(*b*) $q = 30°, p = 50°, r = 100°$.
(*c*) $q = 90°, p = 20°, r = 70°$.
(*d*) $q = 30°, x = 150°$.
(*e*) $p = 60°, r = 130°$.
(*f*) $x = 140°, p = 70°, q = 80°$.
(*g*) $x = 70°, q = 80°$.

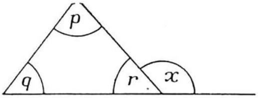


Fig. 94

3. If in Fig. 95 AD is perpendicular to BC and BE is perpendicular to AC,

(*a*) What is the complement of $\widehat{DCA}$?

(*b*) What is the complement of $\widehat{AHE}$?

(*c*) Name *two* complements of $\widehat{HBD}$.

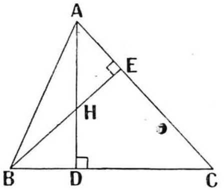


FIG. 95

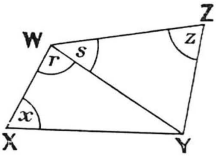


FIG. 96

4. (*a*) Find $\widehat{XYZ}$ in Fig. 96 when

(i) $x = 60°, r = 70°, s = 40°, z = 30°$.
(ii) $x = 100°, r = 20°, s = 70°, z = 80°$.

(*b*) What happens if $x = 20°$, $r = 30°$, $s = 100°$, and $z = 90°$?

5. (*a*) If three angles of a quadrilateral are 40°, 100°, and 160°, what is the fourth angle?

(*b*) Given that the three angles of a quadrilateral are x, y, and z, express the fourth angle in terms of x, y, and z.

6. Find the third angle of a triangle when two of its angles are

(*a*) $x°, 2x°$. (*b*) $x°, 60°$. (*c*) $(2x - 40)°, (30 + x)°$.

7. Find x when the three angles of a triangle are

(*a*) $x°, 2x°, 3x°$. (*b*) $(x + 20)°, (2x + 30)°, (3x + 10)°$.

8. Given that two triangles have two angles of one equal to two angles of the other, prove that their third angles are also equal.

9. Referring to Fig. 97, find the value of y

(*a*) When $p = 40°$, $q = 50°$, and $x = 60°$.
(*b*) In terms of p, q, and x.

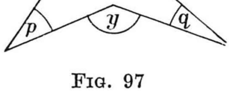


FIG. 97

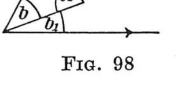


FIG. 98

10. Given in Fig. 98 that $a = a_1$, and $b = b_1$, prove that $x = 90°$.

11. Given in Fig. 99 that $w = 90°$, find the value of v when
(a) $x = 20°$, $y = 30°$. (b) $z = 100°$, $y = x$.

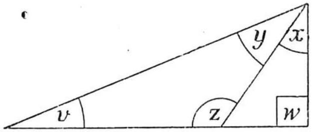


FIG. 99

Section B

12. In Fig. 100 if $a = a_1$ and $b = b_1$, what is the value of x?

13. Given in Fig. 101 that $a = a_1$ and $b = 100°$, find the value of x.

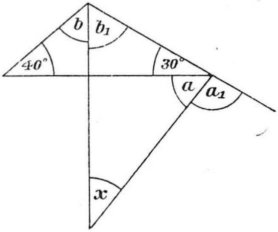


FIG. 100

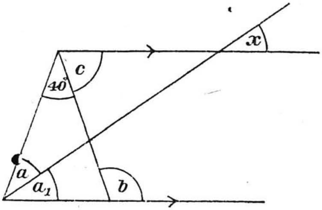


FIG. 101

14. In Fig. 101 if $a = a_1$ and $c = 70°$, what is the value of x?

15. In Fig. 102 $a = a_1$. Find the value of m.

16. Referring to Fig. 103, find the value of $a + b + c$ in degrees.

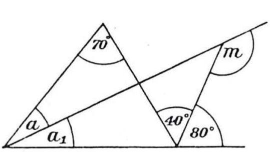


FIG. 102

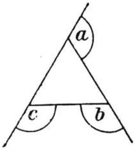


FIG. 103

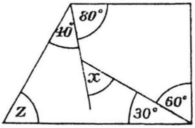


FIG. 104

17. Referring to Fig. 104, find the value of x when $z = 70°$.

18. Find the sum of the exterior angles of a quadrilateral.

19. Find the sum of the interior angles of a convex pentagon.

20. Is the result of the preceding exercise true for the re-entrant pentagon shown in Fig. 105?

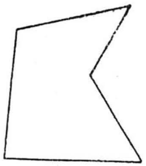


FIG. 105

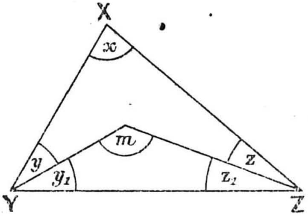


FIG. 106

21. Given that $y = y_1 = 35°$ and $z = z_1 = 25°$ in Fig. 106, prove that $m = 2x$.

22. Given that $y = y_1$ and $z = z_1$ in Fig. 106, prove that $2m - x = 180°$.

23. Find a relation between the angles x, y, a, and b in Fig. 107.

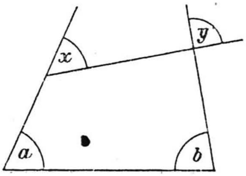


FIG. 107

24. Given in Fig. 108 that $a = a_1$ and $b = b_1$, find a relation between x and y.

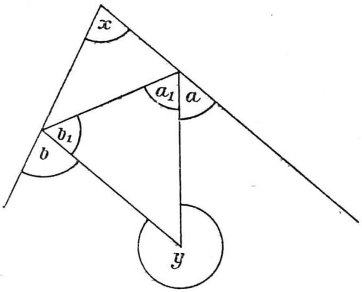


FIG. 108

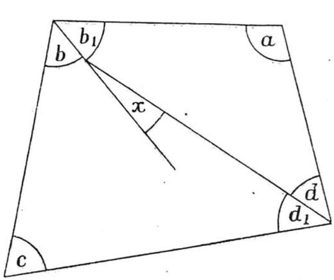


FIG. 109

25. Given in Fig. 109 that $b = b_1$ and $d = d_1$, express x in terms of a and c.

CONGRUENT TRIANGLES

Two triangles are said to be *congruent* if the six elements—*i.e.*, three sides and three angles—of the one triangle are respectively equal to the six elements of the other triangle.

Thus, the triangles in Fig. 110 are congruent if

$$\mathsf{AB} = \mathsf{DE}, \qquad \hat{\mathsf{A}} = \hat{\mathsf{D}},$$
$$\mathsf{BC} = \mathsf{EF}, \qquad \hat{\mathsf{B}} = \hat{\mathsf{E}},$$
$$\mathsf{CA} = \mathsf{FD}, \quad \text{and} \quad \hat{\mathsf{C}} = \hat{\mathsf{F}}.$$

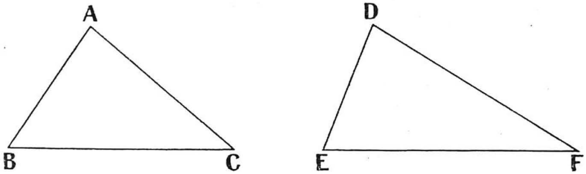


FIG. 110

$\hat{\mathsf{A}}$ is said to be *included* between the sides AB and AC.

The two sides AB in △ ABC and DE in △ DEF are said to correspond. Similarly, BC and EF are corresponding sides. In the same way, $\hat{\mathsf{A}}$ and $\hat{\mathsf{D}}$ are corresponding angles. It should be noted that the word 'corresponding' has a different meaning when we deal with parallel lines. Two corresponding angles in two congruent triangles are opposite to two corresponding sides, and *vice versa*—*e.g.*, $\hat{\mathsf{C}}$ and $\hat{\mathsf{F}}$ are opposite the equal sides AB and DE.

△ ABC ≡ △ DEF means, "triangle ABC is congruent to triangle DEF."

Consideration of the following series of exercises and of those given on pp. 38–40 suggests the truth of the next four theorems.

EXERCISES: XIV

Draw two triangles (non-congruent wherever possible) for each of the following cases:

1. AB = 6 cm., BC = 5 cm., CA = 5·5 cm.
2. BC = 6 cm., $\widehat{B}$ = 50°, $\widehat{C}$ = 70°.
3. AB = 5 cm., BC = 6 cm., $\widehat{B}$ = 47°.
4. AB = 5 cm., BC = 6 cm., $\widehat{A}$ = 90°.
5. $\widehat{A}$ = 30°, $\widehat{B}$ = 80°, $\widehat{C}$ = 70°.
6. AB = 8 cm., BC = 4 cm., $\widehat{A}$ = 30°.
7. AB = 8 cm., BC = 5·5 cm., $\widehat{A}$ = 30°.
8. AB = 8 cm., BC = 8 cm., $\widehat{A}$ = 30°.
9. AB = 8 cm., BC = 9 cm., $\widehat{A}$ = 30°.

THEOREM 2

If two triangles have two sides of the one respectively equal to two sides of the other and the included angle of the one equal to the included angle of the other, then the triangles are congruent.

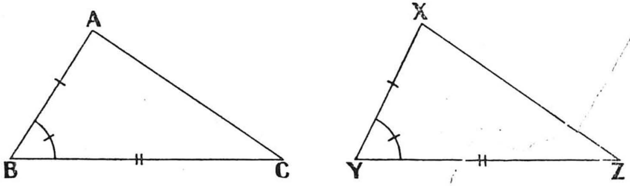


FIG. 111

Given two △s ABC, XYZ, in which

BA = YX,

BC = YZ,

and included angle B = included angle Y.

Prove that △ ABC ≡ △ XYZ.

A 'proof' is given on p. 370.

THEOREM 3

If two triangles have two angles of the one respectively equal to two angles of the other and a side of the one equal to the corresponding side of the other, then the triangles are congruent.

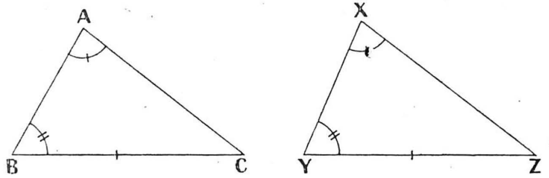


FIG. 112

Given two $\triangle$s ABC, XYZ, in which $\hat{A} = \hat{X}$, $\hat{B} = \hat{Y}$, and the side BC = the corresponding side YZ.

Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

A 'proof' is given on p. 371.

NOTE. The third fact in the data of this theorem might be replaced by AB = XY or by AC = XZ. The point to be noticed is that the sides concerned must correspond.

THEOREM 6

If two triangles have three sides of the one respectively equal to the three sides of the other, then the triangles are congruent.

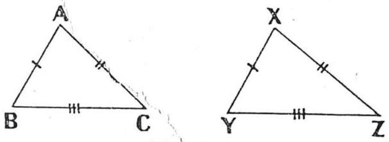


FIG. 113

Given two $\triangle$s ABC, XYZ, in which

AB = XY, BC = YZ, and CA = ZX.

Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

A 'proof' is given on pp. 372–373.

THEOREM 7

If two right-angled triangles have their hypotenuses equal and another side of one triangle equal to a second side of the other triangle, then the triangles are congruent.

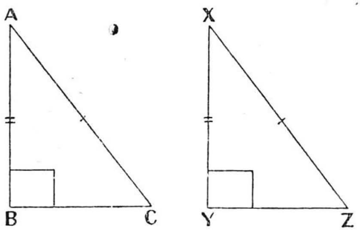


FIG. 114

Given two $\triangle$s ABC, XYZ, in which

$$\text{hypotenuse AC} = \text{hypotenuse XZ},$$
$$AB = XY,$$
$$\text{and } \hat{B} = \hat{Y} = 90°.$$

Prove that $\triangle ABC \equiv \triangle XYZ$.

The proof is given on pp. 374–375.

TWO IMPORTANT CASES

In each of the four preceding theorems, Nos. 2, 3, 6, and 7, three facts are given, and from them the triangles can be proved congruent. In addition to these four cases, two other cases of importance may arise, but in *neither* of these can we infer that the triangles are congruent.

The two other cases are:

(i) If two triangles have three angles of the one respectively equal to three angles of the other, then we *cannot* infer that the triangles are congruent *unless we have some additional information.*

Consider Fig. 115, in which

$$a = a_1,$$
$$b = b_1,$$
$$\text{and } c = c_1.$$

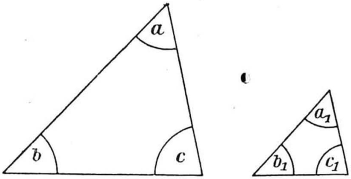


FIG. 115

(ii) If two triangles have two sides of the one equal to two sides of the other and a non-included angle of one is equal to a non-included angle of the other, then we *cannot* infer that the triangles are congruent *unless we have some additional information.*

Consider Fig. 116, in which triangles ABC and XYZ have

$$AB = XY,$$
$$AC = XZ,$$
$$\text{and } \widehat{B} = \widehat{Y}.$$

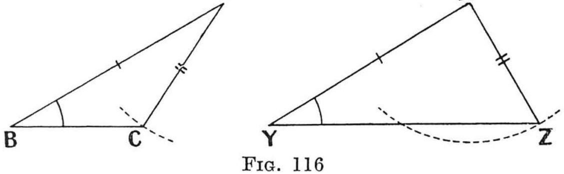


FIG. 116

In this case, although from the data we cannot say the triangles are necessarily congruent, we can prove that either

$$\text{(i) } \triangle ABC \equiv \triangle XYZ,$$
$$\text{or (ii) } \widehat{C} + \widehat{Z} = 180°.$$

(*Cf.* Paper XVI, No. 3, p. 147.)

EXERCISES : XV

1. Draw, in each of the following cases, a pair of triangles ABC and DEF, marking similarly each pair of equal elements. State, with reasons, whether the triangles are necessarily congruent. When the triangles are necessarily congruent state which pairs of remaining elements are equal.

(*a*) AB = DE, BC = EF, CA = FD.

(*b*) $\widehat{A} = \widehat{D}$, $\widehat{B} = \widehat{E}$, $\widehat{C} = \widehat{F}$.

(*c*) AB = DE, AC = DF, $\widehat{A} = \widehat{D}$.

(*d*) AB = DE, AC = DF, $\widehat{B} = \widehat{E}$.

(*e*) $\widehat{A} = \widehat{D}$, $\widehat{B} = \widehat{E}$, AC = DE.

(*f*) $\widehat{A} = \widehat{D}$, $\widehat{C} = \widehat{F}$, BC = EF.

(*g*) AB = DE, AC = EF, $\widehat{C} = \widehat{F} = 90°$.

(*h*) BC = DF, AC = EF, $\widehat{C} = \widehat{F} = 90°$.

2. Given that two straight lines AD and BC bisect each other, prove that AB = CD (see Fig. 117).

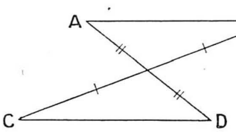


FIG. 117

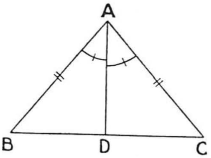


FIG. 118

3. In Fig. 118 the line AD bisects the angle BAC of a △ ABC in which AB = AC. Prove that AD bisects the side BC.

4. Prove that in Fig. 118 $\widehat{ABC} = \widehat{ACB}$.

5. In the △ ABC, AB = AC and D is the mid-point of BC. Prove that (*a*) $\widehat{BAD} = \widehat{CAD}$, (*b*) $\widehat{BDA} = 90°$.

6. In the quadrilateral ABCD, AB = CD and $\widehat{ABC} = \widehat{BCD}$. Prove that AC = BD.

7. Given that any point is taken on the bisector of an angle, prove that the perpendiculars drawn from this point on to the arms of the angle are equal.

8. In the quadrilateral ABCD, BC = CD, $\widehat{ABC} = \widehat{CDA} = 90°$. Prove that CA bisects $\widehat{BAD}$.

9. In the quadrilateral ABCD, $\widehat{ABC} = \widehat{CDA}$ and CA bisects $\widehat{BCD}$. Prove that AB = AD.

10. Prove that the line drawn from the centre of a circle at right angles to any chord bisects that chord. (A chord of a circle is a straight line joining any two points on its circumference.)

11. Prove that the line joining the centre of a circle to the mid-point of any chord is perpendicular to that chord.

12. Given that the line BD bisects the line AC and that AB is parallel to DC, prove that AC bisects BD.

13. In the quadrilateral ABCD, AB = DC and AD = BC. Prove that (*a*) $\widehat{CAB} = \widehat{ACD}$, (*b*) AB ∥ DC, (*c*) AD ∥ BC.

14. In the quadrilateral ABCD, AD = BC and AD ∥ BC. Prove that AB = DC.

15. On the equal sides AB and AC of a △ ABC points P and Q are taken such that AP = AQ. Prove that PC = BQ.

16. Through the point B of a △ ABC a line BE is drawn such that BE is equal and parallel to AC. Through E, ED is drawn parallel to CB to meet AB produced at D. Prove that AB = BD.

17. In the △ PCD, PC = PD. Lines AP and BP are taken outside the triangle such that $\widehat{APC} = \widehat{BPD}$ and each angle is less than $\widehat{CPD}$. If PA = PB prove that AD = BC.

ISOSCELES AND EQUILATERAL TRIANGLES

An *isosceles* triangle is a triangle with two equal sides.

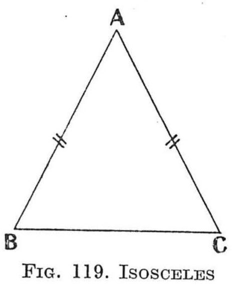


FIG. 119. ISOSCELES

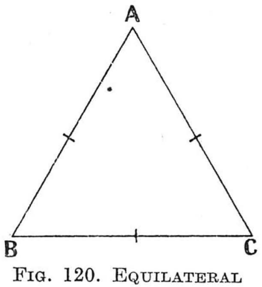


FIG. 120. EQUILATERAL

An *equilateral* triangle is a triangle with three equal sides.

Thus, an equilateral triangle *is also* an isosceles triangle, but an isosceles triangle is *not necessarily* equilateral.

If a triangle has three unequal sides it is a *scalene* triangle.

THEOREM 4

If two sides of a triangle are equal, then the angles opposite these sides are equal.

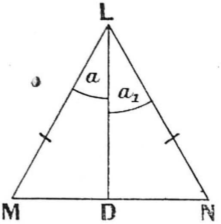


FIG. 121

Given a $\triangle$ LMN in which LM = LN.

Prove that $\widehat{M} = \widehat{N}$.

Construction. Draw LD to bisect $\widehat{MLN}$ and to cut MN at D.

Proof. In the $\triangle$s LMD, LND,

$$\begin{cases} LM = LN & \text{(given)}, \\ a = a_1 & \text{(construction)}, \\ LD = LD & \end{cases}$$

$\therefore \triangle LMD \equiv \triangle LND$ (two sides, included angle), and, in particular,

$\widehat{M} = \widehat{N}$. *Q.E.D.*

THEOREM 5

If in a triangle two angles are equal, then the sides opposite these angles are also equal.

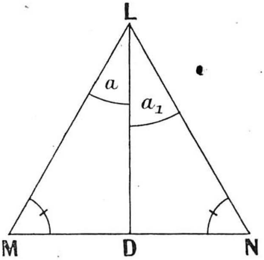


FIG. 122

Given a $\triangle$ LMN with $\widehat{M} = \widehat{N}$.

Prove that LM = LN.

Construction. Draw LD to bisect $\widehat{MLN}$ and meet MN at D.

Proof. In the $\triangle$s LMD, LND,

$$\begin{cases} LD = LD, & \\ \widehat{M} = \widehat{N} & \text{(given),} \\ a = a_1 & \text{(construction).} \end{cases}$$

$\therefore \quad \triangle LMD \equiv \triangle LND$ (two angles, corr. side),

and, in particular,

LM = LN. *Q.E.D.*

NOTE. This is the converse of Theorem 4. Converses are discussed on p. 106.

Construction 6 on p. 132 may now be taken.

EXERCISES: XVI

Section A

1. Find the value of the angle x in each of Figs. 123–131.

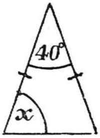


FIG. 123

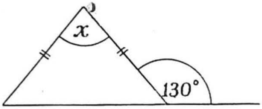


FIG. 124

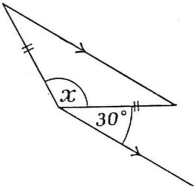


FIG. 125

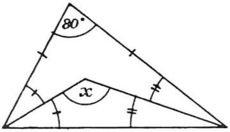


FIG. 126

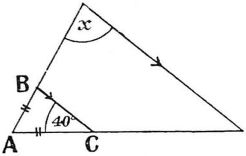


FIG. 127

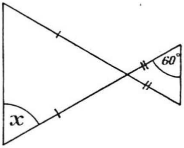


FIG. 128

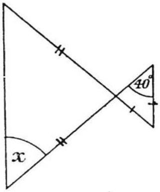


FIG. 129

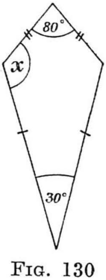


FIG. 130

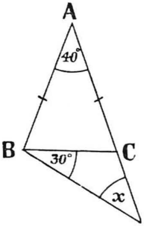


FIG. 131

2. From Fig. 132 give x in terms of y.

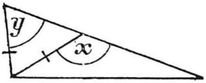


FIG. 132

3. In the $\triangle$ ABC, AB = AC, BI bisects $\widehat{ABC}$, and CI bisects $\widehat{ACB}$. Prove that BI = CI.

4. Given that D is any point on the bisector of $\widehat{BAC}$, and DP is drawn parallel to CA to meet BA at P, prove that PA = PD.

5. In the quad[l] ABCD, AB = AD and CB = CD. Prove that $\widehat{ABC} = \widehat{ADC}$.

6. In the △ ABC, $\widehat{ACB} = 30°$ and P is a point on BC such that BA = AP = PC. Prove that $\widehat{BAC} = 90°$.

7. In Fig. 133 P is the centre of the circle. Prove that $x + y = 2(a + b)$.

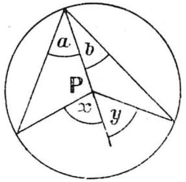


FIG. 133

Section B

8. If in Fig. 134 AB = AC, $r = 50°$, and $s = 15°$, prove that △ ABD is isosceles. (See also Exercise 23.)

9. Given that in Exercise 6 $\widehat{ACB} = x°$, prove that

$$\widehat{BAC} = (180 - 3x)°.$$

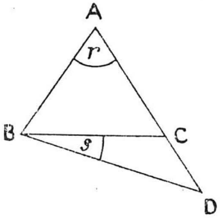


FIG. 134

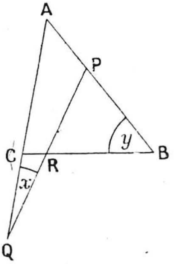


FIG. 135

10. If in Fig. 135 AB = BC and PR = PB, find the value of x when $y = 40°$.

11. If in Fig. 135 PR = PB, $\widehat{A} = 50°$, and $x = 15°$, prove that AB = BC. (See also Exercise 24.)

12. In the quad[l] ABCD, AC cuts BD at X. Given that AX = BX and CX = DX, prove that AB ∥ DC.

13. In the quad[l] ABCD, AB ∥ DC and $\widehat{ADC} = \widehat{DCB}$. Prove that AD = BC.

14. In the △s ABC and XYZ, $\widehat{BAC} = \widehat{YXZ}$, AB = AC, XY = XZ, and BC = YZ. Prove that the triangles are congruent.

15. O is the centre of the circle passing through A, B, and C, $\widehat{BOC} = 100°$, and $\widehat{COA} = 120°$. Find $\widehat{BAC}$, $\widehat{CBA}$, and $\widehat{ACB}$.

16. Given that A is the centre of the circle in Fig. 136, find the value of x when $a = 10°$.

17. Find the value of the angle x in Fig. 136 when a has any value.

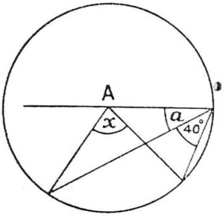


FIG. 136

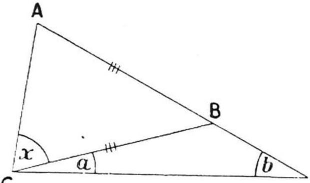


FIG. 137

18. Given that AB = BC in Fig. 137, prove that

$$2x + a + b = 180°.$$

19. The equal sides of an isosceles triangle are AB and AC. Their mid-points are F and E. Prove that FC = BE.

20. In the △ LDB, LD = LB. Lines DA and BC are drawn outside the triangle such that DA = BC and $\widehat{ADL} = \widehat{CBL}$ (each less than $\widehat{LDB}$). Prove that $\widehat{DAB} = \widehat{DCB}$.

21. A point P is taken inside the right angle BAC such that PB = PA = PC. Prove that BPC is a straight line.

22. Prove that any point on the bisector of the vertical angle of an isosceles triangle is equidistant from the ends of the base of the triangle.

23. Given that in Fig. 134 AB = AC and $3r + 2s = 180°$, prove that ABD is isosceles. If AB = AC and if $3r - 2s = 180°$, is the △ ABD now isosceles?

24. Given that in Fig. 135 PR = PB and $2x + 3y = 180°$, prove that △ ABC is isosceles.

INEQUALITIES

Theorem 8

If in a triangle one side is greater than a second side, then the angle opposite the greater side is greater than the angle opposite the smaller side.

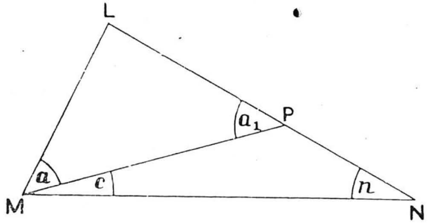


Fig. 138

Given a △ LMN with LN > LM.

Prove that $\widehat{LMN} > \widehat{LNM}$.

Construction. Along LN make LP = LM. Join MP.

Proof. Since LN > LM (given),

∴ P lies between L and N.

∴ $a_1 > n$ (ext. angle of △ MPN).

Also $a_1 = a$ (LM = LP).

∴ $a > n$.

∴ $c + a > n$,

i.e., $\widehat{LMN} > \widehat{LNM}$. *Q.E.D.*

THEOREM 9 (FIRST METHOD)

If in a triangle one angle be greater than a second angle, then the side opposite the greater angle is greater than the side opposite the smaller angle.

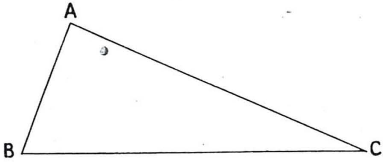


FIG. 139

Given a $\triangle$ ABC with $\widehat{B} > \widehat{C}$.

Prove that AC $>$ AB.

Proof. One of the following three statements must be true:

Either (i) AC $<$ AB, or (ii) AC $=$ AB, or (iii) AC $>$ AB.

(i) Suppose AC $<$ AB,

then $\widehat{B} < \widehat{C}$ (inequality theorem).

But this is contrary to what is given.

$\therefore$ the supposition is wrong.

$\therefore$ AC is not less than AB.

(ii) Suppose AC $=$ AB,

then $\widehat{B} = \widehat{C}$ (isosceles triangle theorem).

But this is contrary to what is given.

$\therefore$ the supposition is wrong.

$\therefore$ AC is not equal to AB.

Since AC is neither less than AB nor equal to AB,

$\therefore$ AC $>$ AB. *Q.E.D.*

The method of this proof consists of examining all the possibilities and of proving that all but one are wrong. It is called an *indirect* method, or a *proof by exhaustion*.

THEOREM 9 (SECOND METHOD)

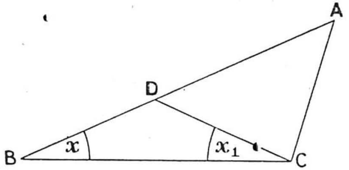


FIG. 140

Given a $\triangle$ ABC with $\widehat{ACB} > \widehat{ABC}$.

Prove that AB > AC.

Construction. From the larger angle BCA construct the angle BCD equal to x.

Proof. Since $\widehat{ACB} > x$ (given),

$\therefore$ CD lies inside the $\triangle$ ACB.

$\therefore$ D lies between B and A.

Since AC is the shortest path from A to C,*

$\therefore$ AD + DC > AC.

Also DC = DB $(x = x_1)$.

$\therefore$ AD + DB > AC,

i.e., AB > AC. *Q.E.D.*

* See Assumption 2, p. 53.

The Distance of a Point from a Line

In Figs. 141 and 142 PX is the perpendicular from the point P to the straight line AB, produced if necessary. The

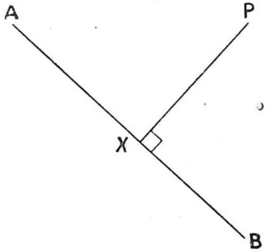


Fig. 141

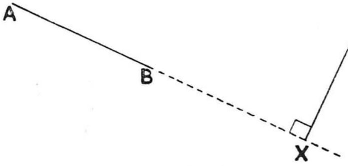


Fig. 142

length of PX is commonly called the *distance* of P from AB.

Equidistant, Concurrent, Collinear

A point P is said to be *equidistant* from two straight lines AB and CD when the perpendicular distances PX and PY are equal (see Fig. 143).

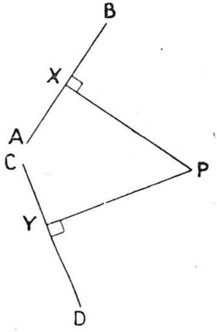


Fig. 143

If three straight lines meet in a point they are said to be *concurrent*.

If three points lie on the same straight line they are said to be *collinear*.

THEOREM 10

If a point be taken outside a given straight line, then of all the lines that may be drawn from this point to the given line the perpendicular is the shortest.

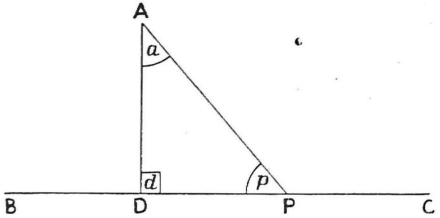


FIG. 144

Given a straight line BC with A a point outside it. AD is perpendicular to BC, and AP is *any* other line from A to BC.

Prove that AD is less than AP.

Proof. Since

$a + d + p = 2$ right angles (sum of angles of a triangle),

and $d = 1$ right angle (given),

$\therefore \quad a + p = 1$ right angle,

$\therefore \quad a + p = d.$

$\therefore \quad d > p,$

$\therefore$ AP > AD (inequality theorem),

i.e., AD is less than AP. *Q.E.D.*

EXERCISES: XVII

Section A

1. In Fig. 145 which is greater, AB or BC?

2. In Fig. 146 which is greater:

(*a*) BD or AD?
(*b*) AB or CD?

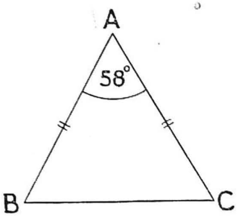


FIG. 145

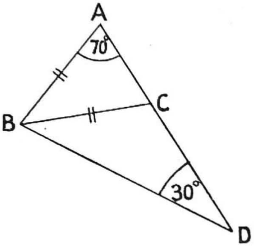


FIG. 146

3. In Fig. 147 which is greater, AC or CD?

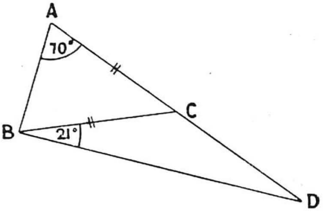


FIG. 147

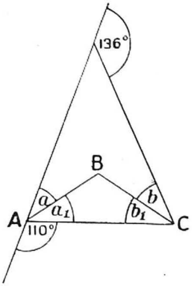


FIG. 148

4. If in Fig. 148 $a = a_1$ and $b = b_1$, which is greater, AB or BC?

5. In the triangle ABC, BC $>$ BA and the bisectors of angles BAC and BCA meet at I. Prove that CI $>$ AI.

6. Given that K is any point inside the $\triangle$ ABC, prove that $\widehat{BKA} > \widehat{BCA}$. (*Hint.* Produce AK and use the corollary to Theorem 1.)

7. A point P is taken on the side BC of an equilateral triangle ABC. Prove that AB $>$ AP $>$ BP.

8. A point P is taken on the base BC of a triangle whose sides AB and AC are equal. Prove that AB > AP.

9. A point P is taken inside the △ ABC. Prove that

$$PB + PC < AB + AC.$$

(*Hint.* Produce BP to cut AC, and use the fact that the sum of two sides of a triangle is greater than the third side. See note to Exercise 20.)

10. Use the preceding exercise to prove that

$$AB + BC + CA > PA + PB + PC > \frac{AB + BC + CA}{2}.$$

11. Prove that in △ ABC, AB + AC > 2AD, where AD is the median through A. (*Hint.* Produce AD to K, making DK = AD.)

Section B

12. In Fig. 149 which is greater, AB or CD?

13. Prove that in Fig. 150 AB > FE. (*Hint.* Show that AB > BF > FD > FE.)

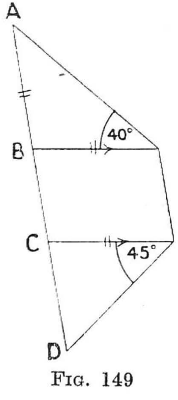


FIG. 149

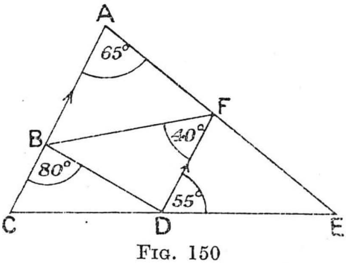


FIG. 150

14. In the quad[l] ABCD, AB ∥ DC and the diagonals cut at E. Given that BE > AE, prove that ED > EC.

15. In the △ ABC, $\widehat{ABC} = 90°$ and $\widehat{ACB} = 32°$. Prove that 2AB > AC.

16. In the △ ABC, AB > AC and the perpendiculars from B to AC and C to AB meet at H. Prove that HB > HC.

17. Put AB, BC, and CA of Fig. 151 in order of size when $a = a_1$ and $b = b_1 = 30°$.

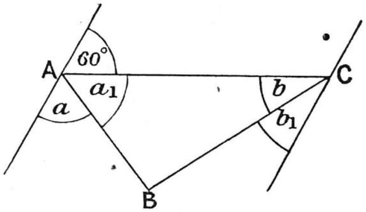


FIG. 151

18. In the quad[1] ABCD the longest side is CD and the shortest AB. Prove that (*a*) $\widehat{ABC} > \widehat{ADC}$; (*b*) $\widehat{ABC} + \widehat{BAD} > 2$ right angles.

19. In Fig. 152 which is greater, AC or AB?

20. The bisector of the angle ACB of a $\triangle$ ABC meets AB at D. Prove that AC > AD and CB > DB.

(NOTE. *It follows that* AC + CB > AB. This result is often assumed as a fundamental property of a straight line—*i.e.*, that the shortest journey from A to B is made by travelling along the straight line AB.)

FIG. 152

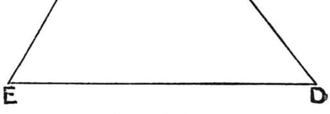


FIG. 153

21. Use the result of the previous exercise to prove that in Fig. 153 AE + ED + DA > BC + CD + DE + EB.

22. Given that O is any point inside the quad[1] ABCD, prove that OA + OB + OC + OD > AC + BD for all positions of O except one. Which one?

PARALLELOGRAMS

Definitions. A quadrilateral which has one pair of parallel sides is called a **trapezium** (Fig. 154).

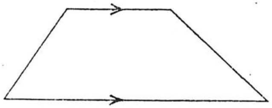


FIG. 154. TRAPEZIUM

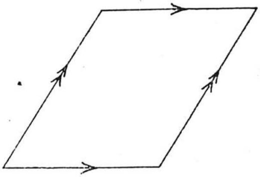


FIG. 155. PARALLELOGRAM

A quadrilateral which has both pairs of **opposite** sides parallel is called a **parallelogram** (Fig. 155).

A parallelogram with a pair of adjacent sides equal is called a **rhombus** (Fig. 156).

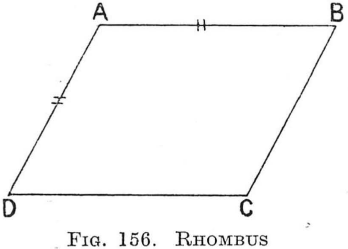


FIG. 156. RHOMBUS

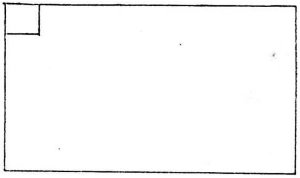


FIG. 157. RECTANGLE

A parallelogram in which one angle is a right angle is called a **rectangle** (Fig. 157).

A rectangle with a pair of adjacent sides equal is called a **square** (Fig. 158).

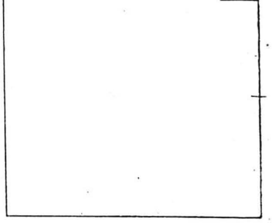


FIG. 158. SQUARE

THEOREM 11

In a parallelogram the opposite sides are equal and the opposite angles are equal.

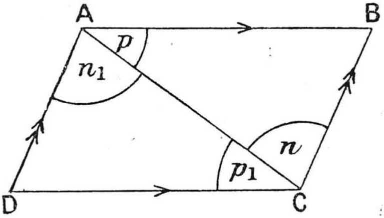


FIG. 159

Given a parallelogram ABCD.

Prove that (i) AB = DC and BC = AD;

(ii) $\widehat{B} = \widehat{D}$ and $\widehat{BAD} = \widehat{DCB}$.

Construction. Draw the diagonal AC.

Proof. In the Δs ABC, CDA,

$$\begin{cases} p = p_1 & \text{(alt. angles; AB} \parallel \text{DC)}, \\ n = n_1 & \text{(alt. angles; AD} \parallel \text{BC)}, \\ \text{AC} = \text{CA}. & \end{cases}$$

$\therefore$ Δ ABC $\equiv$ Δ CDA (two angles, corr. side),

and, in particular,

(i) AB = DC and BC = AD. *Q.E.D.*

(ii) $\widehat{B} = \widehat{D}$. *Q.E.D.*

Also $p = p_1$ (proved)

and $n_1 = n$ (proved).

$\therefore$ $p + n_1 = p_1 + n$,

i.e., $\widehat{BAD} = \widehat{DCB}$. *Q.E.D.*

Corollary. A parallelogram is divided into two triangles of equal area by a diagonal.

THEOREM 12

In a parallelogram the diagonals bisect each other.

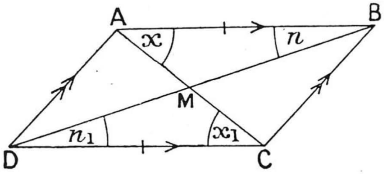


FIG. 160

Given a parallelogram ABCD whose diagonals, AC and BD, meet at M.

Prove that AM = CM
and BM = DM.

Proof. In the △s AMB, CMD,

$$\begin{cases} x = x_1 & \text{(alt. angles; AB} \parallel \text{DC),} \\ n = n_1 & \text{(alt. angles; AB} \parallel \text{DC),} \\ \text{AB} = \text{CD} & \text{(parallelogram theorem).} \end{cases}$$

∴ △ AMB ≡ △ CMD (two angles, corr. side),

and, in particular,

AM = CM
and BM = DM. *Q.E.D.*

THEOREM 13

If one pair of opposite sides of a quadrilateral are equal and parallel, then the second pair of opposite sides are equal and parallel.

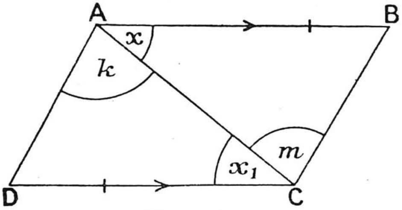


FIG. 161

Given a quadrilateral ABCD with AB = DC and AB ∥ DC.

Prove that (i) BC = AD,

(ii) BC ∥ AD.

Construction. Join AC.

Proof. In the △s ABC, CDA,

$$\begin{cases} AB = CD & \text{(given)}, \\ AC = CA, & \\ x = x_1 & \text{(alt. angles; } AB \parallel DC). \end{cases}$$

∴ △ ABC ≡ △ CDA (two sides, included angle),
and, in particular,

(i) BC = AD. *Q.E.D.*

(ii) $m = k$.

Since these are equal alternate angles,

∴ BC ∥ DA. *Q.E.D.*

NOTE. From the definition of a parallelogram it follows that ABCD is a parallelogram.

THEOREM 14

If the opposite sides of a quadrilateral are equal the quadrilateral is a parallelogram.

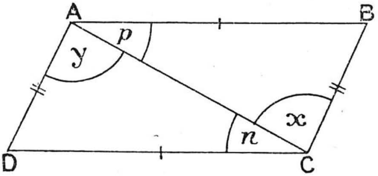


FIG. 162

Given a quadrilateral ABCD in which AB = DC and BC = AD.

Prove that ABCD is a parallelogram.

Construction. Join AC.

Proof. In the $\triangle$s ABC, CDA,

$$\begin{cases} AB = CD & \text{(given)}, \\ BC = AD & \text{(given)}, \\ CA = AC. \end{cases}$$

$\therefore$ $\triangle$ ABC $\equiv$ $\triangle$ CDA (three sides),

and, in particular,

$$p = n.$$

Since these are equal alternate angles,

$\therefore$ AB $\parallel$ DC.

Again, $x = y$ (from the congruent triangles).

Since these also are equal alternate angles,

$\therefore$ BC $\parallel$ AD.

Since the opposite sides of the quadrilateral are parallel,

$\therefore$ ABCD is a parallelogram. *Q.E.D.*

NOTE. Theorems 14, 15, and 16 are the converse theorems of Theorems 11 and 12. Converses are discussed on p. 106.

THEOREM 15

If the opposite angles of a quadrilateral are equal the quadrilateral is a parallelogram.

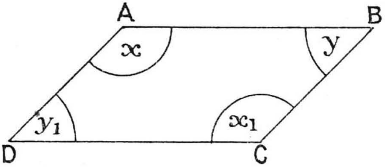


FIG. 163

Given a quadrilateral ABCD with $x = x_1$ and $y = y_1$.

Prove that ABCD is a parallelogram.

Proof. We know that

$$x + y + x_1 + y_1 = 360° \quad \text{(sum of angles of a quad}^{\text{l}}\text{)},$$
$$x = x_1,$$
$$\text{and} \quad y = y_1 \quad \text{(given)}.$$
$$\therefore \quad 2x + 2y = 360°,$$
$$\therefore \quad x + y = 180°.$$

Since x and y are supplementary interior angles,

$\therefore$ AD || BC.

Similarly, we may prove that

AB || DC.

Therefore ABCD is a quadrilateral whose opposite sides are parallel—*i.e.*, it is a parallelogram. *Q.E.D.*

THEOREM 16

If the diagonals of a quadrilateral bisect each other the quadrilateral is a parallelogram.

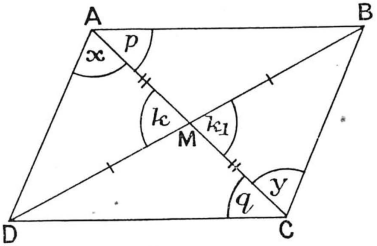


FIG. 164

Given a quadrilateral ABCD whose diagonals bisect each other at M.

Prove that ABCD is a parallelogram.

Proof. In the $\triangle$s AMD, CMB,

$$\begin{cases} \text{AM} = \text{CM} & \text{(given)}, \\ \text{DM} = \text{BM} & \text{(given)}, \\ k = k_1 & \text{(vertically opposite)}. \end{cases}$$

$\therefore \quad \triangle \text{AMD} \equiv \triangle \text{CMB}$ (two sides, included angle),
and, in particular,

$$x = y.$$

Since these are alternate angles,

$$\therefore \quad \text{AD} \parallel \text{BC}.$$

Similarly, by taking $\triangle$s AMB, CMD, we can prove that the alternate angles p and q are equal.

$$\therefore \quad \text{AB} \parallel \text{DC}.$$

Since ABCD is a quadrilateral with its opposite sides parallel,

$\therefore$ ABCD is a parallelogram. *Q.E.D.*

Construction 7 on p. 133 may now be taken.

Important Properties. Just as from the definition of a parallelogram a number of important properties of the parallelogram were proved, so we may prove properties of the rhombus, rectangle, and square from their respective definitions (see p. 98). Many of these properties are given in the following set of exercises:

Exercises: XVIII

Prove the following theorems:

Theorem 16*a*. All the sides of a rhombus are equal.

Theorem 16*b*. The diagonals of a rhombus bisect each other at right angles.

Theorem 16*c*. Each diagonal of a rhombus bisects the angle of the rhombus through which it passes.

Theorem 16*d*. Each angle of a rectangle is a right angle.

Theorem 16*e*. The diagonals of a rectangle are equal.

(Note that the five theorems 16*a* to 16*e* are also true of the square.)

Theorem 16*f*. A quadrilateral whose four sides are equal is a rhombus.

Theorem 16*g*. A quadrilateral whose diagonals bisect each other at right angles is a rhombus.

Theorem 16*h*. If in a parallelogram one diagonal bisects one of the angles the parallelogram is a rhombus.

Theorem 16*i*. If each angle of a quadrilateral is a right angle the quadrilateral is a rectangle.

Theorem 16*j*. If the diagonals of a parallelogram are equal the parallelogram is a rectangle.

CONVERSE THEOREMS

Consider the following two theorems:

(i) If in the $\triangle$ ABC, AB $=$ AC, then $\hat{B} = \hat{C}$.

(ii) If in the $\triangle$ ABC, $\hat{B} = \hat{C}$, then AB $=$ AC.

In the first theorem

the datum is AB $=$ AC, and the conclusion is $\hat{B} = \hat{C}$.

In the second theorem

the datum is $\hat{B} = \hat{C}$, and the conclusion is AB $=$ AC.

When two theorems are related in this way each is said to be the *converse* of the other.

In the above example a theorem is given whose converse theorem is true, and we have met other examples of theorems whose converse theorems are true—*e.g.*, Theorems 8 and 12.

Some theorems are such that their converse theorems are not true. An example is:

If a quadrilateral is a square, then its diagonals are equal.

The converse theorem is:

If a quadrilateral has its diagonals equal, then the quadrilateral is a square. This is *not* a true theorem.

(See also Exercise 3 in Paper VI, p. 144.)

Exercises: XIX (*on Theorems* **11–16**)

Section A

1. One angle of a parallelogram is 70°. Calculate the other angles.

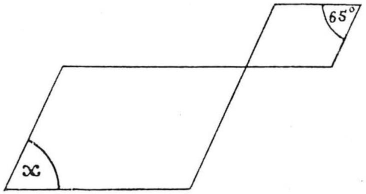


Fig. 165

2. Fig. 165 is made up of two parallelograms. Find the value of x.

3. Given that in Fig. 166 AB = AC, find the value of x.

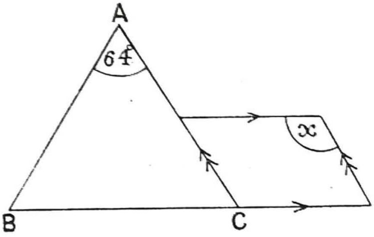


FIG. 166

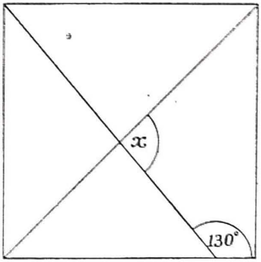


FIG. 167

4. What is the value of the angle x in the square shown in Fig. 167.

5. The angle between the diagonals of a rectangle is 80°. Find the angle between a diagonal and a long side.

6. The angle between a diagonal of a rhombus and a side is 20°. Calculate the angles of the rhombus.

7. In a parallelogram ABCD, AL is the perpendicular from A to BD and CM is the perpendicular from C to BD. Prove that ALCM is a parallelogram.

8. The two parallelograms ABCD and ABXY have a common base AB and DYCX is a straight line. Prove that the △s DYA and BXC are congruent.

9. Given that ABCD in Fig. 168 is a trapezium and r and s are right angles, prove that XY = PQ.

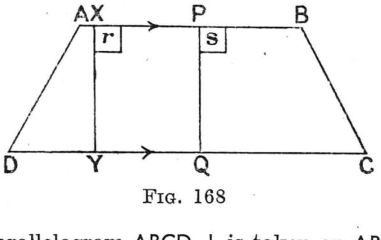


FIG. 168

10. In the parallelogram ABCD, L is taken on AB and M on DC such that AL = CM. Prove that ALCM is a parallelogram.

11. If AC and LM in the previous exercise cut at O prove that DOB is a straight line.

12. Three parallelograms are drawn as in Fig. 169. Prove that the length of the path from A to D along the sides *via* X, Y, and Z is the same as the length of the path from A to D along the sides *via* B and C.

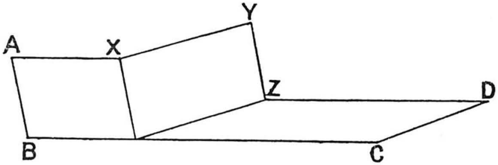


FIG. 169

13. Draw a quadrilateral ABCD which is *not* a parallelogram, having a pair of opposite angles each 40° and a pair of opposite sides each 3 in. long.

14. The bisectors of the angles of a parallelogram form the sides of a quadrilateral ABCD. Prove that AC = BD. (*Hint.* Use Theorem 16*e*.)

Section B

15. Find the value of the angle x in the parallelogram shown in Fig. 170.

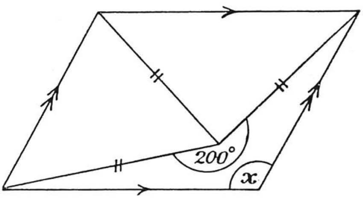


FIG. 170

16. If in Fig. 169 $\widehat{BAX} = 79°$ and $\widehat{XYZ} = 86°$, what is the value of $\widehat{CDZ}$?

17. State three properties of a square which are not, in general, true of a parallelogram.

18. In the parallelogram ABCD points L and M are taken on BD such that BM = DL. Prove that $\widehat{MCL} = \widehat{MAL}$.

19. In the parallelogram ABCD the diagonals meet at O. A line LOM is drawn cutting one pair of opposite sides at L and M. A line ROS is drawn cutting the other pair of sides at R and S. Prove that LSMR is a parallelogram.

20. The side AB of the parallelogram ABCD is produced to E so that AB = BE. Prove that DE bisects BC.

21. Two parallelograms ABXY and CDYX are drawn on the same base XY. Prove that ABCD is a parallelogram.

22. Two circles are drawn having the same centre S. Two straight lines are drawn through S. The first line cuts one circle in P and Q, and the second line cuts the other circle in L and M. Prove that PLQM is a parallelogram.

23. In the parallelogram ABCD, $\widehat{BAD}$ is obtuse. Prove that BD > AC.

24. The side AB of the parallelogram ABCD is produced both ways to P and Q so that PA = AB = BQ. Given that DA = 2AB, prove that DQ is perpendicular to CP.

INTERCEPT

Definition. If the straight line x cuts the straight lines a and b at P and Q, then the portion of x lying between

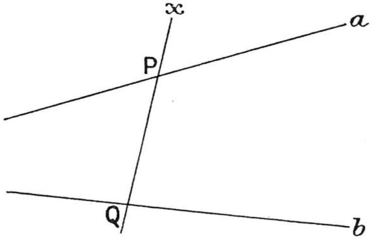


FIG. 171

P and Q is called the *intercept* which a and b cut off along x (Fig. 171).

INTERCEPT THEOREMS

THEOREM 17

If in any triangle a line be drawn through the mid-point of one side parallel to a second side this line will bisect the third side.

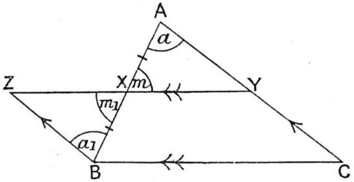


FIG. 172

Given a △ ABC with AX = XB and XY ∥ BC.

Prove that AY = CY.

Construction. Draw BZ ∥ CA to meet YX produced at Z.

Proof. In the △s AXY, BXZ,

$$\begin{cases} a = a_1 & \text{(alt. angles; BZ} \parallel \text{CA),} \\ m = m_1 & \text{(vertically opposite),} \\ \text{AX} = \text{BX} & \text{(given).} \end{cases}$$

∴ △ AXY ≡ △ BXZ (two angles, corr. side),

and, in particular,

AY = BZ.

Since ZY ∥ BC (given),

and ZB ∥ YC (construction),

∴ ZYCB is a parallelogram.

∴ BZ = CY (opposite sides of a parallelogram),

and BZ = AY (proved).

∴ AY = CY. *Q.E.D.*

THEOREM 18

The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.

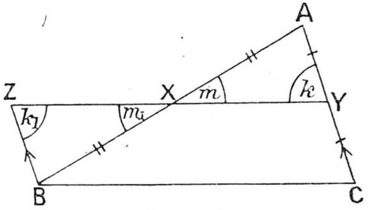


FIG. 173

Given the $\triangle$ ABC with AX = BX and AY = CY.

Prove that (i) XY $\parallel$ BC,

(ii) XY = $\frac{1}{2}$BC.

Construction. Draw through B a line BZ $\parallel$ CA to meet YX produced in Z.

Proof. (i) In the $\triangle$s AXY, BXZ,

$$\begin{cases} AX = BX & \text{(given)}, \\ m = m_1 & \text{(vert. opp.)}, \\ k = k_1 & \text{(alt. angles; CA} \parallel \text{BZ).} \end{cases}$$

$\therefore$ $\triangle$ AXY $\equiv$ $\triangle$ BXZ (two angles, corr. side), and, in particular,

AY = BZ I

and YX = ZX II

Since AY = CY (given),

$\therefore$ BZ = CY (from I).

Also BZ $\parallel$ CY (construction).

$\therefore$ BCYZ is a parallelogram (BZ and CY equal and parallel).

$\therefore$ ZY $\parallel$ BC,

i.e., XY $\parallel$ BC. *Q.E.D.*

(ii) Since ZY = BC (from the par^{m} BCYZ),

and XY = $\frac{1}{2}$ZY (from II),

$\therefore$ XY = $\frac{1}{2}$BC. *Q.E.D.*

Theorem 19

If three or more parallel straight lines cut off equal intercepts on one straight line which cuts them, then they cut off equal intercepts on any other straight line which cuts them.

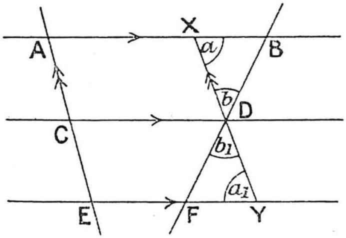


Fig. 174

Given three parallel lines AB, CD, and EF cut by ACE and BDF, and AC = CE.

Prove that BD = FD.

Construction. Draw through D a line XDY parallel to AE to meet AB, EF, at X, Y.

Proof. By definition, AXDC is a parallelogram.

∴ AC = XD (opposite sides of a parallelogram).

Similarly, CE = DY (CDYE a parallelogram).

Since AC = CE (given),

∴ XD = DY.

In the △s XDB, YDF,

XD = DY (proved),

$a = a_1$ (alt. angles; AB ∥ EF),

$b = b_1$ (vertically opposite).

∴ △ XDB ≡ △ YDF (two angles, corr. side),

and, in particular,

BD = FD. *Q.E.D.*

Note 1. This proof may be easily extended to cover the case of more than three parallel lines.

NOTE 2. Theorem 17 is a special case of Theorem 19.

If E and F of Fig. 174 coincide, as shown in Fig. 175, we get a △ AEB with C the mid-point of AE and CD ∥ AB. Theorem 19 shows that BD = ED, as has already been proved in Theorem 17.

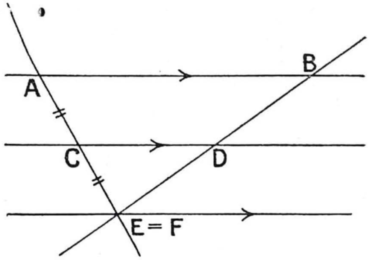


FIG. 175

EXERCISES: XX

Section A

1. Calculate the length x in Fig. 176.

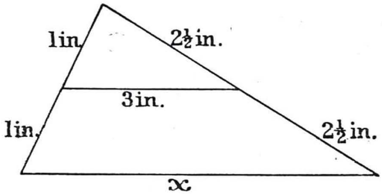


FIG. 176

2. Given that L and M are the mid-points of the sides AB and AC of a △ ABC and P is any point on BC, prove that LM bisects AP.

3. The mid-points of the sides of a △ ABC are X, Y, and Z. Prove that △s ABC and XYZ are equiangular.

4. Given that in Fig. 177 AB = 4 in., calculate the length of AC.

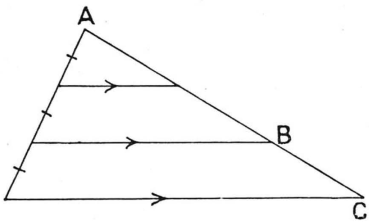


FIG. 177

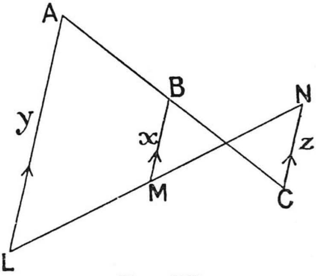


FIG. 178

5. In Fig. 178 AB = BC and LM = MN. Calculate the value of x when $y = 5$ in. and $z = 2$ in.

6. Prove that the figure formed by joining the mid-points of the sides of a quadrilateral is a parallelogram.

7. Prove that in Fig. 174 AB + EF = 2CD.

8. Prove that the lengths x, y, and z in Exercise 5 are such that $y \sim z = 2x$.

Section B

9. Given that in Fig. 179 AB = 3 in., BC = 5 in., and AD = 4 in., calculate the length of AE.

10. Given that in the △ ABC, L, M, and N are the mid-points of BC, CA, and AB respectively, prove that AL bisects MN.

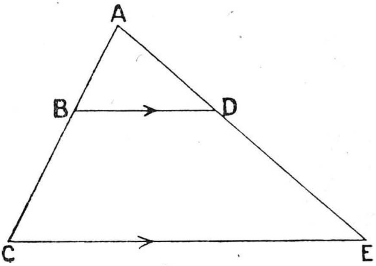


FIG. 179

11. Prove that in the diagram constructed in the previous exercise the area of △ AMN equals a quarter of the area of the △ ABC.

12. Given that the diagonals of a quadrilateral are perpendicular, prove that the parallelogram formed by joining the mid-points of the sides is a rectangle.

13. If the diagonals of a quadrilateral are equal prove that the parallelogram formed by joining the mid-points of the sides is a rhombus.

14. If the diagonals of a quadrilateral are equal and are perpendicular, what sort of figure is the parallelogram formed by joining the mid-points of the sides?

15. Given that in the △ ABC, X is the mid-point of AC and $\widehat{ABC} = 90°$, prove that BX = $\frac{1}{2}$AC. (*Hint.* Draw XL perp^{r} to BC.)

16. Prove that in the par^{m} ABCD the line joining B to the mid-point of AD trisects AC. (*Hint.* Join D to the mid-point of BC.)

17. A perpendicular AK is drawn from A on to the bisector of the angle ACB of a △ ABC. Prove that the line drawn through K parallel to CB bisects AB.

18. The medians BE and CF of the △ ABC cut at G. From B draw BK parallel to FC to cut AG produced at K. Prove (*a*) that KC is parallel to BE; (*b*) that **the three medians of a triangle meet in a point, which point trisects each median.** This point is called the **centroid** of the triangle.

ANGLES OF A POLYGON

THEOREM 20

In a convex polygon of *n* sides the sum of the interior angles is (2*n* — 4) right angles.

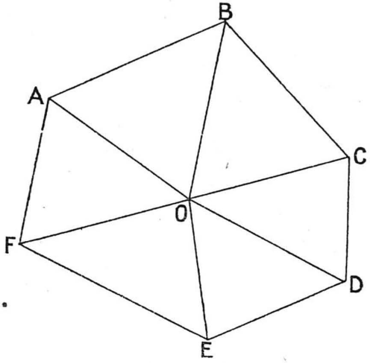


FIG. 180

Given a convex polygon ABCD . . . having n sides.

Prove that $\widehat{ABC} + \widehat{BCD} + \ldots = (2n - 4)$ rt. angles.

Construction. Join any point O inside the polygon to A, B, C, etc. (Fig. 180 shows a polygon in which $n = 6$.)

Proof. There are n triangles, such as $\triangle$ AOB, $\triangle$ BOC, etc.

The sum of the three angles of each triangle = 2 rt. angles.

$\therefore$ the sum of all the angles of the n triangles = $2n$ rt. angles.

Now, this latter sum is made up of

$\widehat{AOB} + \widehat{BOC} + \ldots (= X$, say)

and $\widehat{ABC} + \widehat{BCD} + \ldots (= Y$, say).

$\therefore \quad X + Y = 2n$ right angles,

and $X = 4$ right angles (complete revolution).

$\therefore Y = (2n - 4)$ right angles.

i.e., $\widehat{ABC} + \widehat{BCD} + \ldots = (2n - 4)$ right angles.

Q.E.D.

THEOREM 21

If at each vertex of a convex polygon of *n* sides an exterior angle be formed the sum of these exterior angles is 4 right angles.

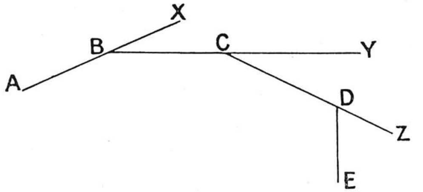


FIG. 181

Given a polygon ABCDE . . . having n sides which are produced to X, Y, Z, etc., respectively.

Prove that

$$\widehat{XBC} + \widehat{YCD} + \widehat{ZDE} + \ldots = 4 \text{ right angles.}$$

Proof. The sum of an interior and an exterior angle at each vertex is 2 right angles.

Let P denote the sum of the n exterior angles
and Q ,, ,, ,, ,, ,, ,, interior ,,

Then $P + Q = 2n$ right angles (since there are n vertices).

Also $Q = (2n - 4)$ right angles (interior angles of polygon).

$\therefore$ $P = 4$ right angles.

That is,

$$\widehat{XBC} + \widehat{YCD} + \widehat{ZDE} + \ldots = 4 \text{ right angles.}$$

Q.E.D.

EXERCISES: XXI

1. Find the fifth angle of a pentagon when the other four angles are:

(*a*) 100°, 105°, 120°, 125°.
(*b*) Each 100°.

2. Find the sixth angle of a hexagon when the other five angles are:

(*a*) 110°, 120°, 130°, 140°, 150°.
(*b*) 123°, 72°, 81°, 142°, 133°.

3. Write down the number of degrees in each angle of a regular (*a*) pentagon, (*b*) hexagon, (*c*) heptagon, (*d*) octagon, (*e*) fifteen-sided polygon.

4. Write down the number of degrees in each exterior angle of each of the figures given in the previous exercise.

5. State in which of the following cases a regular polygon can be drawn:

(*a*) Each exterior angle is (i) 40°, (ii) 57°, (iii) 60°, (iv) 65°, (v) 180°.
(*b*) Each interior angle is (i) 100°, (ii) 108°, (iii) 110°, (iv) 117°, (v) 160°.

6. If the five angles of a pentagon are $x°$, $(x - 20)°$, $(x + 30)°$, $(2x - 80)°$, and $(x + 10)°$, what is the value of x?

7. In Fig. 182, . . ABCD . . is part of a regular hexagon and . . EBCF . . is part of a regular octagon. Calculate the value of x.

8. ABCDEF is a regular hexagon. Calculate the value of $\widehat{BAC}$ and $\widehat{CAD}$.

9. In a heptagon three of the exterior angles are equal, and each of the other four exterior angles is 5° greater than each of the first three. Find each exterior angle.

10. Show that in the regular pentagon ABCDE the bisector of $\widehat{AEB}$ is perpendicular to DE.

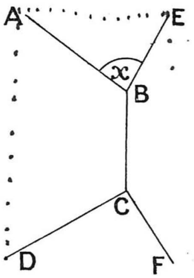


FIG. 182

11. Given that in the regular octagon ABCDEFGH, AD meets BF at X, prove that △ ABX is isosceles.

12. In the pentagon ABCDE the sides AE and CD are parallel, and the angles at A, B, and C are equal. Calculate the size of $\widehat{B}$.

13. If in the preceding exercise $\widehat{A}$ is also equal to $\widehat{E}$ prove that BC and ED are parallel.

14. Fig. 183 shows the opened-out surfaces of a pyramid on a regular pentagon ABCDE as base. The $\triangle$s AXB, BYC, etc., are all

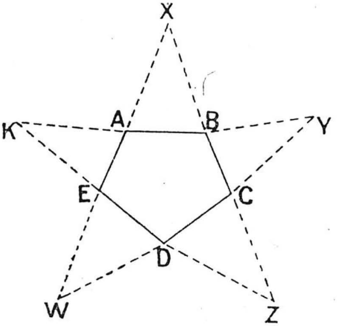


FIG. 183

isosceles. Given that $\widehat{AXB} = 40°$, calculate the value of $\widehat{XBY}$. For such a pyramid to be constructed, what is the greatest possible value of $\widehat{AXB}$?

15. In the regular pentagon ABCDE the bisector of $\widehat{CAE}$ meets CE at X. Prove that BX is perpendicular to AC.

SYMMETRY

Fold a piece of paper about any straight line AB.

From the folded paper cut out any figure—*e.g.*, the shaded figure in Fig. 184. Now unfold the paper, and a hole will be

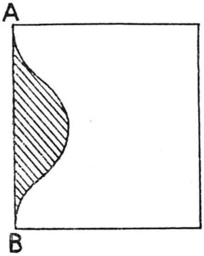


FIG. 184

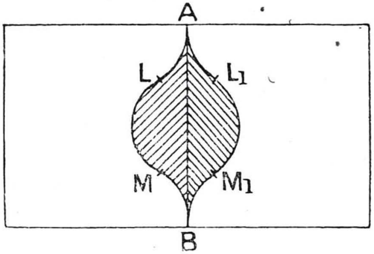


FIG. 185

obtained, as shown by the shaded portion of Fig. 185. This shaded figure possesses the property that the part of the figure on one side of AB can be turned about AB so as to coincide with the part on the other side of AB. Such a figure is called a *symmetrical figure*, and AB is called an *axis of symmetry*.

If when the folding is done L fits on L_1, then L and L_1 are called *corresponding points*, or L_1 is said to be the *image* of L.

There are two important properties of a symmetrical figure, namely, if L, L_1 and M, M_1 are any two pairs of corresponding points:

(i) LL_1 is bisected at right angles by the line AB.

(ii) LM is equal to L_1M_1.

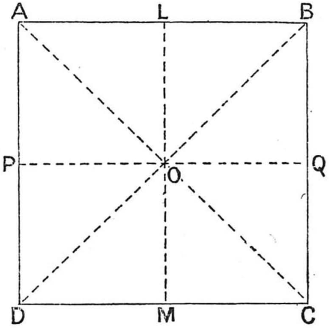


FIG. 186

Some figures have more than one axis of symmetry. For example, the square in Fig. 186 is symmetrical about each of the four axes AC, BD, LM, and PQ.

Consider the parallelogram ABCD in Fig. 187. No axis of symmetry exists.

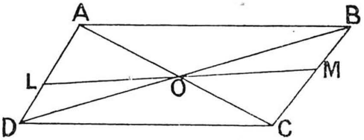


FIG. 187

The point O is such that every straight line through O and bounded by the parallelogram is bisected at O—*e.g.*,

$$LO = OM.$$

The parallelogram is said to be *symmetrical about the point* O, and O is called the *centre of symmetry*.

EXERCISES: XXII

Are the following figures symmetrical? If so, state the axis (or axes) of symmetry and (or) the centre of symmetry.

1. An isosceles triangle.
2. An equilateral triangle.
3. A trapezium.
4. A regular pentagon.
5. A circle.
6. Two equal intersecting circles.
7. Two unequal intersecting circles.
8. A rhombus.
9. The letter S.
10. The letter L.
11. The letter E.
12. The letter Z.

CONSTRUCTIONS

We have already drawn figures with the aid of a graduated ruler and a protractor. We now come to a new type of construction, for which we are allowed to use only an ungraduated straight-edge and compasses.

CONSTRUCTION 1

To construct a line which passes through a given point in a given straight line and makes with that line an angle equal to a given angle.

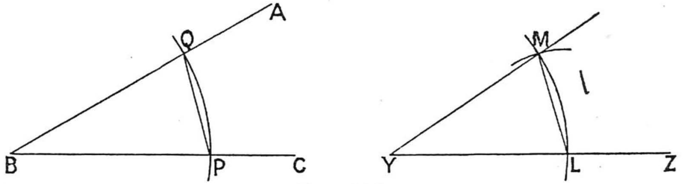


FIG. 188

Given an angle ABC and a straight line YZ.

Construct a line through Y which makes with YZ an angle equal to $\widehat{B}$.

Construction. With centre B and any radius draw a circle to cut BC at P and BA at Q.

With centre Y and the *same* radius draw a circle to cut YZ at L.

With centre L and radius equal to PQ draw a circle to cut the second circle at M.

Then YM is the required line.

Proof. In the $\triangle$s BPQ, YLM,

$$\begin{cases} BP = YL & \text{(radii of equal circles)}, \\ BQ = YM & \text{(radii of equal circles)}, \\ PQ = LM & \text{(construction)}. \end{cases}$$

$$\therefore \quad \triangle BPQ \equiv \triangle YLM \quad \text{(three sides)},$$

and, in particular,

$$\widehat{B} = \widehat{Y},$$

i.e., MY makes with YZ an angle equal to $\widehat{B}$.

NOTE. In all constructions any necessary construction lines should be clearly shown.

Exercises: XXIII

1. Use Construction 1 to copy the following angles drawn with the aid of a protractor: (*a*) 70°; (*b*) 100°.

2. Draw any angle. Copy it, using Construction 1, and check by measurement.

3. Draw any acute angle ABC. Construct a line XB such that $\widehat{XBC} = 3\widehat{ABC}$.

4. Draw a large quadrilateral ABCD at random. With the aid of a straight-edge and compasses only, copy the quadrilateral by making lines and angles equal to AB, $\widehat{DAB}$, $\widehat{ABC}$, BC, and $\widehat{BCD}$ in the order given. Check by measuring CD.

5. Draw any circle. Take A, C, D, and B any four points (in this order) on the circumference. Without using a protractor, make $\widehat{ADX} = \widehat{ACB}$, with X and C on opposite sides of AD. Does DX pass through B?

6. Draw any line AB. Take any point C not on AB. Join CA. Without using a protractor, make $\widehat{ACD}$ alternate to $\widehat{BAC}$ and equal to $\widehat{BAC}$. (*Notice that you have made* CD *parallel to* BA.)

7. Draw any line AB. Take any point C not on AB. Join CA and produce to X. Without using a protractor, make $\widehat{XCD}$ corresponding to $\widehat{XAB}$ and equal to $\widehat{XAB}$. (*Notice that you have made* CD *parallel to* AB.)

8. Draw any line AB. Take any point P not on AB. Join P to any point Q on AB. On the opposite side of AB from P, without using a protractor, draw $\widehat{RQB}$ equal to $\widehat{PQB}$. With compass point on Q and radius QP, draw a circle to cut QR at S. Join PS. (*Notice that you have made* PS *perpendicular to* AB.)

Construction 2

To bisect a given angle.

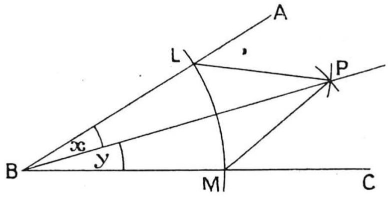


Fig. 189

Given an angle ABC.

Construct a line bisecting $\widehat{ABC}$.

Construction. With centre B and any radius draw a circle to cut BA, BC, at L and M respectively.

With centres L and M and the same radius, which must be greater than $\frac{1}{2}$LM, draw two circles to intersect at P.

Join BP.

Then BP is the required bisector.

Join PL, PM.

Proof. In the △s BLP, BMP,

$$\begin{cases} BL = BM & \text{(radii of same circle),} \\ LP = MP & \text{(radii of equal circles),} \\ PB = PB. & \end{cases}$$

$$\therefore \triangle BLP \equiv \triangle BMP \quad \text{(three sides),}$$

and, in particular,

$$x = y,$$

i.e., BP bisects $\widehat{ABC}$.

EXERCISES: XXIV

1. Which is the axis of symmetry in Fig. 189?

2. With a protractor make angles of (*a*) 40°, (*b*) 90°, (*c*) 174°. Without further use of the protractor, bisect each of these angles.

3. Draw any obtuse angle, and bisect it without using a protractor. Measure the two halves to the nearest degree.

4. Draw a triangle of sides 6 cm., 7 cm., and 8 cm. Without using a protractor, bisect the three angles. Do the three bisectors meet in a point?

5. Repeat the previous exercise for any triangle. Do the bisectors meet in a point?

6. Without using a protractor bisect the four angles of any large quadrilateral. Do the four bisectors meet in a point?

7. Draw an angle just less than 90°. Without using a protractor, divide it into four equal parts.

8. Explain why Construction 2 would enable us to find $\frac{1}{32}$ of a given angle, but not $\frac{1}{33}$ of the angle.

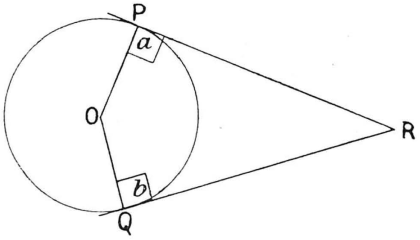


FIG. 190

9. Copy Fig. 190, making your circle 2 in. in radius. Without using a protractor, bisect $\widehat{PRQ}$. Does the bisecting line pass through O?

10. Draw any △ ABC and produce AB to D and AC to E. By using Construction 2 bisect $\widehat{BAC}$, $\widehat{DBC}$, and $\widehat{BCE}$. Do the bisectors meet in a point?

CONSTRUCTION 3

To construct a straight line at right angles to a given unlimited straight line from a given point on the line.

First Method

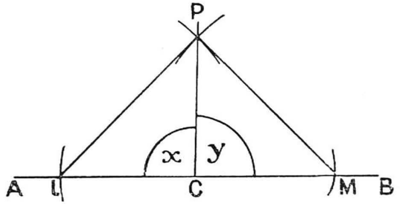


FIG. 191

Given an unlimited straight line AB with a point C on it.

Construct a line through C perpendicular to AB.

Construction. With centre C and any radius, draw a circle to cut AB at L and M.

With centres L and M and equal radii ($> \frac{1}{2}$LM) draw two circles to cut at P.

Join CP.

Then CP is the required perpendicular.

Proof. In the $\triangle$s LPC, MPC,

$$\begin{cases} \text{LC} = \text{MC} & \text{(radii of same circle)}, \\ \text{PL} = \text{PM} & \text{(radii of equal circles)}, \\ \text{PC} = \text{PC}. \end{cases}$$

$$\therefore \quad \triangle \text{LPC} \equiv \triangle \text{MPC} \quad \text{(three sides)},$$

and, in particular,

$$x = y.$$

$$\text{Since} \quad x + y = 180^\circ \quad \text{(half a revolution)},$$

$$\therefore \quad x = 90^\circ.$$

$\therefore$ CP is a line through C perpendicular to AB.

Second Method

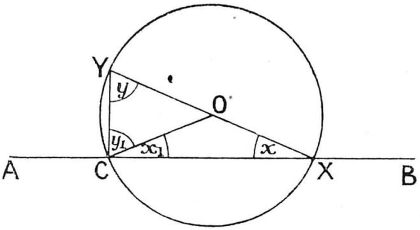


FIG. 192

Given an unlimited straight line AB with a point C on it.

Construct a line through C perpendicular to AB.

Construction. Take any point O outside AB.

Draw the circle with centre O and radius OC to cut AB again at X.

Join XO and produce to meet the circle at Y.

Join CY.

Then CY is the required perpendicular.

Proof. We have $x_1 = x$ (OX = OC),

and $y_1 = y$ (OY = OC).

$\therefore \quad x_1 + y_1 = x + y.$

$\therefore \quad x_1 + y_1 = \frac{1}{2}(x + y + x_1 + y_1).$

But $x + y + x_1 + y_1 = 180°$ (sum of angles of $\triangle$ YCX),

$\therefore \quad x_1 + y_1 = 90°,$

i.e., CY is a line through C perpendicular to AB.

NOTE. A shorter proof follows from Theorem 36.

CONSTRUCTION 4

To construct a straight line to bisect a given limited straight line.

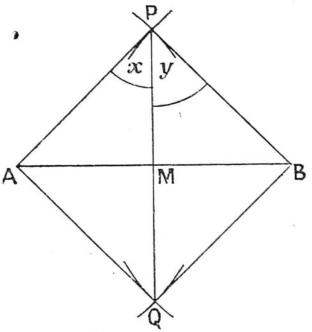


FIG. 193

Given a limited straight line AB.

Construct a straight line to bisect AB.

Construction. With centres A and B and a radius greater than $\frac{1}{2}$AB, draw two equal circles to cut at P and Q. Join PQ.

Then PQ is the required line.

Join PA, PB, QA, and QB. Let PQ meet AB at M.

Proof. In the $\triangle$s APQ, BPQ,

$$\begin{cases} AP = BP & \text{(radii of equal circles)}, \\ AQ = BQ & \text{(radii of equal circles)}, \\ PQ = PQ. \end{cases}$$

$\therefore \quad \triangle\,APQ \equiv \triangle\,BPQ$ (three sides),

and, in particular,

$$x = y.$$

In the $\triangle$s APM, BPM,

$$\begin{cases} AP = BP & \text{(radii of equal circles)}, \\ x = y & \text{(just proved)}, \\ PM = PM. \end{cases}$$

$\therefore \quad \triangle\,APM \equiv \triangle\,BPM$ (two sides, included angle),

and, in particular,

$$AM = BM.$$

$\therefore$ AB is bisected by PQ.

NOTE. The pupil should prove $\widehat{PMA} = 90°$. PQ is called the *perpendicular* or *right bisector* of AB.

EXERCISES: XXV (*on Construction* **3**)

1. Which is the axis of symmetry in Fig. 191?

2. Without using a protractor make a right-angled triangle whose short sides are 3 in. and 4 in. long. Measure the longest side.

3. Without using a protractor make an angle of (*a*) 45°, (*b*) $22\frac{1}{2}$°, (*c*) 135°, (*d*) $157\frac{1}{2}$°.

4. Draw a circle, centre O and radius 2 in. Take any point P on the circumference and, without using a protractor, construct $\widehat{OPQ} = 90°$. Produce QP to R. In how many points does QR meet the circle?

5. Take any point B on the circumference of a circle of which AC is a diameter. Without using a protractor, make $\widehat{ABD} = 90°$. Does BD, produced if necessary, pass through C?

EXERCISES: XXVI (*on Construction* **4**).

1. Which are the axes of symmetry in Fig. 193?

2. Draw a line 9 cm. long. Use Construction 4 to divide the line into four equal parts. Check by measurement.

3. Draw a triangle ABC with AB = 6 cm., AC = 7 cm., and BC = 8 cm. Using Construction 4, bisect AB and AC at P and Q. Measure $\widehat{APQ}$ and $\widehat{ABC}$. Are they equal?

4. Draw a △ ABC with AB = 4 in., $\widehat{BAC} = 70°$, and $\widehat{ABC} = 52°$. Construct, with the aid of compasses and straight-edge, the three right bisectors of the sides. Do they meet in a point?

5. Using Construction 4, draw the right bisectors of the sides of (*a*) any acute-angled triangle, (*b*) any obtuse-angled triangle. Do the right bisectors meet in a point in each case?

6. Draw any quadrilateral. Construct geometrically the four right bisectors of the sides. Do they meet in a point?

7. Using Construction 4, draw the right bisectors of the sides of a quadrilateral whose vertices lie on the circumference of a circle. Do the four right bisectors meet in a point?

8. Suggest a method requiring the use of straight-edge and compasses only of bisecting a line about 13 in. long if the compasses cannot be used to draw a circle of radius more than 4 in.

9. Write out, with proof, the following method of performing Construction 4. Draw circles with centres A and B and the same radius to cut at X. Then the bisector of $\widehat{AXB}$ bisects AB.

CONSTRUCTION 5

To construct a perpendicular to an unlimited straight line from a point outside the line.

First Method

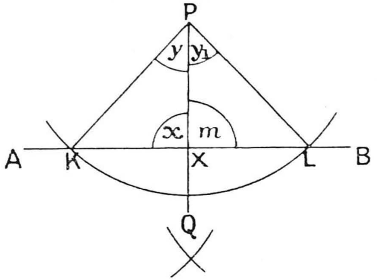


FIG. 194

Given an unlimited straight line AB, with P a point outside it.

Construct from P a line perpendicular to AB.

Construction. Draw a circle with centre P to cut AB at L and K. Join LP, KP. Bisect $\widehat{LPK}$ (known construction), and let this bisector PQ meet AB at X.

Then PX is the required perpendicular to AB.

Proof. In the $\triangle$s PKX, PLX,

$$\begin{cases} PK = PL & \text{(radii of same circle)}, \\ PX = PX, & \\ y = y_1 & \text{(construction)}. \end{cases}$$

$\therefore \quad \triangle PKX \equiv \triangle PLX$ (two sides, included angle),

and, in particular,

$$x = m.$$

Also $x + m = 180°$ (half a revolution).

$\therefore \quad x = 90°.$

Hence PX is perpendicular to AB.

Second Method

Given an unlimited straight line AB, with P a point outside it.

Construct from P a line perpendicular to AB.

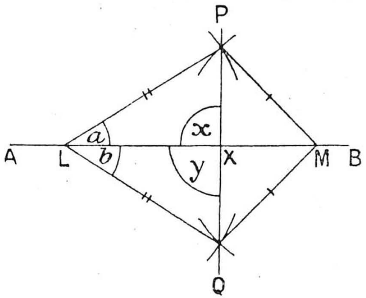


FIG. 195

Construction. Take any two points L and M on AB. With centres L and M and radii LP and MP respectively, draw circles to cut at P and Q.

Join PQ and let it cut AB at X.

Then PQ is the required perpendicular.

Join PL, PM, QL, and QM.

Proof. In the $\triangle$s PLM, QLM,

$$\begin{cases} PL = QL & \text{(radii of same circle)}, \\ PM = QM & \text{(radii of same circle)}, \\ LM = LM. \end{cases}$$

$$\therefore \triangle PLM \equiv \triangle QLM \quad \text{(three sides)},$$

and, in particular,

$$a = b.$$

In the $\triangle$s PLX, QLX,

$$\begin{cases} PL = QL & \text{(radii of same circle)}, \\ LX = LX, \\ a = b & \text{(just proved)}. \end{cases}$$

$$\therefore \triangle PLX \equiv \triangle QLX \quad \text{(two sides, included angle)},$$

and, in particular,

$$x = y.$$

Since $x + y = 180°$ (half a revolution),

$$\therefore \quad x = 90°.$$

Hence PQ is perpendicular to AB.

Third Method

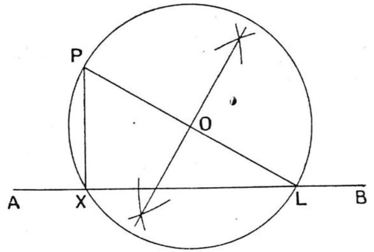


FIG. 196

Construction. Let L be any point on AB. Bisect PL at O. With centre O and radius OP draw a circle to cut AB at L and X. Join PX. Then PX is the required perpendicular.

Proof. The proof is left to the pupil. (*Cf.* p. 217.)

EXERCISES: XXVII

1. Which is the axis of symmetry in (*a*) Fig. 194, (*b*) Fig. 195?

2. Draw a △ ABC with AB = 6 cm., BC = 7 cm., and AC = 8 cm. Construct a perpendicular from A to BC.

3. Construct a perpendicular from A to BC in a triangle ABC in which AB = 4 cm., AC = 8 cm., and BC = 6 cm.

4. Make an angle BAC equal to 83°. Use Construction 2 to draw AX bisecting $\widehat{BAC}$. On AX mark AK = 3 in. Use Construction 5 to draw perpendiculars from K on to AB and AC. Verify by measurement that these perpendiculars are equal.

5. Draw any acute-angled triangle. Construct the three perpendiculars from the vertices on to the opposite sides. Do they meet in a point?

6. Construct the three perpendiculars from the vertices of any obtuse-angled triangle to the opposite sides. Do the three perpendiculars meet in a point?

7. Draw a circle, centre O and radius $2\frac{1}{2}$ in. Construct two chords of the circle each 2 in. long. Construct perpendiculars from O on to the chords, and measure their lengths.

8. Draw any circle. Let AC be a diameter. Draw any line AB through A to cut the circle at D. Construct the perpendicular from C to AB. Verify that this perpendicular passes through D.

CONSTRUCTION 6

To construct an angle of 60° without using a protractor.

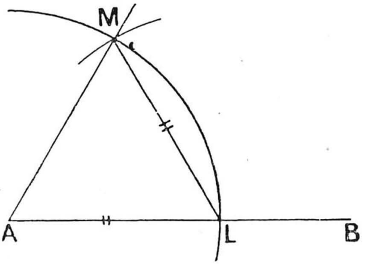


FIG. 197

Construct an angle of 60°.

Construction. Draw any line AB. With centre A and any radius, draw a circle to cut AB at L. With centre L and the same radius, draw a circle to cut the first circle at M. Join AM.

Then $\widehat{A}$ is the required angle.
Join LM.

Proof. Since AL = AM (radii of same circle)
and AL = LM (radii of equal circles),
∴ AL = LM = AM.
∴ △ ALM is equilateral.
∴ $\widehat{A} = 60°$,
i.e., $\widehat{A}$ is the required angle of 60°.

EXERCISES: XXVIII

No protractor should be used for the following exercises.

1. Construct an angle of (*a*) 120°, (*b*) 30°, (*c*) 90°, (*d*) 75°, (*e*) $37\frac{1}{2}°$, and (*f*) $142\frac{1}{2}°$.

2. Construct an isosceles triangle in which the sum of the base angles is five times the value of the vertical angle.

Construction 7

To construct through a given point a straight line parallel to a given straight line.

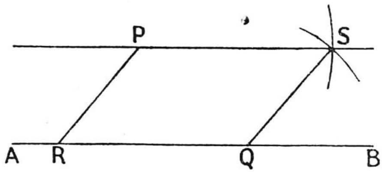


Fig. 198

Given a straight line AB with P a point outside it.

Construct a line through P parallel to AB.

Construction. Take any two points R and Q on AB.

With centre P, radius RQ, and centre Q, radius RP, draw circles.

Let the circles meet at S (on the side of PQ remote from R).

Join PS.

Then PS is the required parallel.

Proof. Since PS = RQ (construction)

and PR = SQ (construction),

$\therefore$ PRQS is a parallelogram (converse of par^{m} theorem).

$\therefore$ PS $\parallel$ RQ.

Hence PS is the line through P parallel to AB.

Compare also the methods indicated in Exercises 6 and 7 on p. 122.

CONSTRUCTION 8

To show how to divide a given straight line into a given number of equal parts.

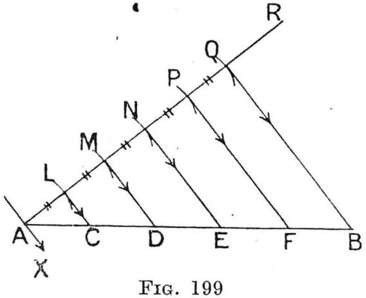


FIG. 199

For this construction we shall suppose the given number of parts to be five. A similar construction will hold for any other integral number.

Given a straight line AB.

Divide AB into five equal parts.

Construction. Draw through A any line AR. With compasses mark off AL = LM = MN = NP = PQ along AR.

Join QB.

Draw PF, NE, MD, LC, and AX each parallel to QB.

Then AB is divided into five equal parts, at C, D, E, and F.

Proof. Since AX, LC, MD, NE, PF, and QB are all parallel to one another, and the intercepts (AL, LM, etc.) made on AR are equal (construction),

$\therefore$ the intercepts on AB are equal (Intercept Theorem),

i.e., AC = CD = DE = EF = FB.

Hence AB is divided into five equal parts by the points C, D, E, and F.

EXERCISES: XXIX (*on Construction* 7)

1. Draw a $\triangle$ ABC with AB = 8 cm., $\widehat{BAC} = 70°$, and $\widehat{ABC} = 45°$. Through C draw a parallel to AB (*a*) using a set-square, (*b*) using a protractor, (*c*) using ruler and compasses only.

2. Draw any straight line AB. Draw a line parallel to AB and 2 in. from AB. (In this numerical example a protractor may be used if desired.)

3. Draw any line AB. Construct a line parallel to AB at a given distance from AB. (For this general construction only a straight-edge and compasses should be used.)

4. Draw a parallelogram with sides 2 in. and 3 in. in length and one angle of 73°. (Numerical; compare instructions to Exercise 2.)

5. Construct a parallelogram with given sides and a given angle. (General; compare instructions to Exercise 3.)

6. Draw a rectangle with sides 5 in. and 3 in. long.

7. Construct a square on a given base.

8. Draw a trapezium ABCD in which AB = 5 in., AD = 2 in., DC = 4 in., $\widehat{CDA} = 80°$, and DC is parallel to AB.

9. Construct a rhombus on a given base and having a given angle.

10. Draw a large circle with centre O. Take any three points A, B, and C on the circle. Through C construct a parallel to AB to meet the circle at D. Let X and Y be the mid-points of AB and CD. Join XY. Does XY pass through O? (Use a straight-edge and compasses only.)

EXERCISES: XXX (*on Construction* 8)

1. Draw a straight line 5 in. long.

(*a*) Divide the line into three equal parts, using a set-square and ruler to draw parallels. No lengths are to be measured.

(*b*) Divide the line into three equal parts, using straight-edge and compasses only.

2. Draw a straight line 4 in. long. Divide it into five equal parts. (A set-square may be used, but no lengths should be measured.)

3. Show how to construct a line whose length shall be one and a half times the length of a given line.

4. Show how to construct a line whose length shall be 2·4 times the length of a given line.

5. Draw a line $4\frac{1}{2}$ in. long. Without further measurement construct a line whose length is $\frac{3}{7}$ of the length of the given line.

6. Using straight-edge, set-square, and compasses only, construct a rectangle one of whose sides is three-quarters the length of the other side.

7. Show how to construct a square, given its perimeter, using straight-edge and compasses only.

EXAMPLE. Draw an angle AOB = 70°, and draw a straight line PQ, 3 in. long, having its ends on AO and BO and making with BO an angle of 43°.

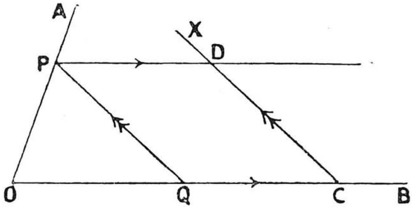


FIG. 200

Order of steps:

(i) Make $\widehat{AOB} = 70°$.
(ii) Take C any point on OB.
(iii) Make $\widehat{OCX} = 43°$.
(iv) On CX mark off CD = 3 in.
(v) Draw DP ∥ BO.
(vi) Draw PQ ∥ DC.

Then PQ is the required line.

If P may be taken on AO produced, and Q may be taken on BO produced, how many solutions can be found?

EXAMPLE. Show how to construct a trapezium given two parallel sides, a third side, and the angle which the fourth side makes with one of the parallel sides.

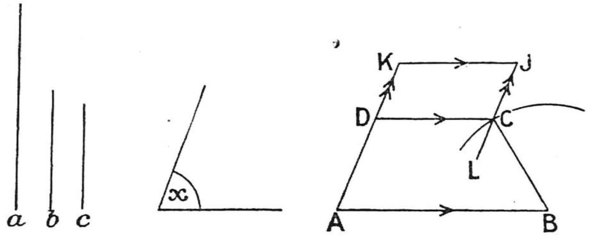


FIG. 201

Given three straight lines a, b, c and an angle x.

Construct a trapezium ABCD with AB $= a$, DC $= b$, BC $= c$, $\widehat{\text{A}} = x$, and DC $\parallel$ AB.

Construction. Make AB $= a$.

Construct $\widehat{\text{KAB}} = x$, KJ $\parallel$ AB and equal to b, and JL $\parallel$ KA.
Draw a circle with centre B and radius $= c$.
Let JL cut this circle at C.
Draw CD $\parallel$ AB.

Then ABCD is the required trapezium.

Proof. Since KJCD is a parallelogram,

$\therefore$ DC = KJ (opp. sides theorem).

$\therefore$ DC $= b$.

Also AB $= a$,

$\widehat{\text{A}} = x$,

and BC $= c$.

$\therefore$ ABCD is the required trapezium.

NOTE. Fig. 202 shows that there are several possibilities with the previous construction. If b is of such a length that J falls at

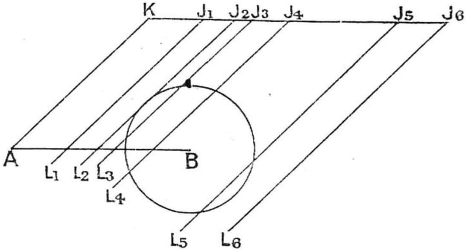


FIG. 202

J_1 or J_6 no trapezium satisfying the conditions can be drawn.

If the length of b is such that J falls on J_3, then J_3L_3 cuts the circle in two points on the same side of AB as J_3, and two trapeziums can be drawn to satisfy the given conditions. How many solutions are there if the length of b is such that (i) J falls on J_2, (ii) J falls on J_4, (iii) J falls on J_5?

EXAMPLE. Show how to construct a triangle with given perimeter and given angles.

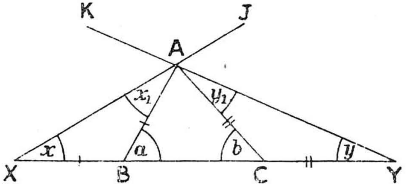


FIG. 203

Analysis. Let △ ABC be the required triangle (Fig. 203). Produce BC both ways, making XB = BA and CY = CA. Thus, XY equals the given perimeter.

We have $x = x_1$ (AB = XB, construction)

$= \frac{1}{2}a$ (a is ext. angle of △ ABX).

Similarly $y = \frac{1}{2}b$.

Since both a and b are given angles, the values of x and y can be calculated.

This analysis suggests the following construction:

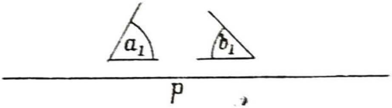


FIG. 204

Given the angles a_1 and b_1 and a straight line p (Fig. 204).

Construct a $\triangle$ ABC with $\widehat{ABC} = a_1$ and $\widehat{ACB} = b_1$, and AB + BC + CA $= p$.

Construction. Make XY $= p$, $\widehat{JXY} = \frac{1}{2}a_1$, and $\widehat{KYX} = \frac{1}{2}b_1$.

Let KY meet JX at A.

Make $\widehat{XAB} = \frac{1}{2}a_1$ and $\widehat{YAC} = \frac{1}{2}b_1$.

Then ABC is the required triangle.

Proof. By construction,

$$x = x_1 \quad (\text{each} = \tfrac{1}{2}a_1).$$

$$\therefore \quad AB = BX.$$

$$\text{Similarly,} \quad AC = CY.$$

$$\begin{aligned} \therefore \quad BC + AB + CA &= BC + BX + CY \\ &= XY \\ &= p. \end{aligned}$$

Also, $a = x + x_1$ (ext. angle of $\triangle$ ABX).

$\therefore \quad a = a_1 \qquad (x = x_1 = \frac{1}{2}a_1)$.

Similarly, $b = b_1$.

$\therefore$ $\triangle$ ABC has the correct perimeter and has

$$\widehat{ABC} = a_1$$

and $\widehat{BCA} = b_1$,

i.e., ABC is the required triangle.

MISCELLANEOUS CONSTRUCTIONS

EXERCISES: XXXI

Except where special instructions are given, proceed as follows in the exercises below. Where numerical measures of lines and angles are given, use a ruler and protractor and give, without proof, the steps of your construction. Where no measures are given, use only a straight-edge and compasses, catalogue your steps, and give a proof.

1. Through a given point on a given straight line construct a straight line which shall make with the given line an angle equal to (*a*) half a given angle, (*b*) the complement of a given angle, (*c*) the supplement of a given angle.

2. Through a given point construct a straight line which makes a given angle with a given straight line which does not pass through the point.

3. On a given straight line as diagonal construct a rhombus with a side of given length.

4. On a given straight line as diagonal construct a square.

5. Through a given point P within an angle BAC construct a line which cuts AB and AC at X and Y so that AX = AY.

6. On a given straight line as base construct an isosceles triangle of given perimeter.

7. Construct an isosceles triangle of given perimeter in which each side is twice the base.

8. When a, b, c, and h have the meaning indicated in Fig. 205,

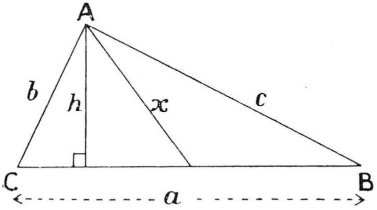


FIG. 205

in which x is a median, draw triangles having the following measurements:

(i) $a = 10$ cm., $b = 7$ cm., $x = 6$ cm.

(ii) $a = 9$ cm., $\widehat{ACB} = 43°$, $h = 6$ cm.

(iii) $a = 9$ cm., $x = 7$ cm., $h = 6$ cm.

(iv) $b = 3$ in., $c = 4$ in., $h = 2{\cdot}5$ in.

(v) $a = 2$ in., $b + c = 4\frac{1}{2}$ in., $\widehat{\mathsf{ACB}} = 80°$.

(vi) $\widehat{\mathsf{ACB}} = 60°$, $\widehat{\mathsf{ABC}} = 65°$, $b + c = 5$ in.

(vii) $a = 7$ cm., $\widehat{\mathsf{ACB}} = 40°$, $b - c = 1$ cm.

9. Construct an isosceles triangle, given the base and the length of the perpendicular from the vertex on to the base.

10. Construct an isosceles triangle, given the vertical angle and the length of the perpendicular from the vertex on to the base.

11. Draw a parallelogram with sides 3 in. and 4 in. long and one angle of 60°.

12. Draw a parallelogram with sides 5 in. and 4 in. and the perpendicular distance between the longer sides 3 in.

13. Draw a parallelogram with diagonals 10 cm. and 8 cm. long and the angle between them 38°.

14. Draw a parallelogram with the longer sides 4 in., the perpendicular distance between them 2 in., and the shorter diagonal $3\frac{1}{2}$ in.

15. Draw a trapezium with the shorter of the parallel sides 3 cm., the perpendicular distance between the parallel sides 4 cm., and the non-parallel sides 5 cm. and 6 cm.

16. Draw a trapezium with one of the parallel sides 3 in., the non-parallel sides 2 in. and $2\frac{1}{2}$ in., and the shorter diagonal $3\frac{1}{2}$ in.

17. Calculate the exterior angle of a regular pentagon. Draw a regular pentagon with 2-inch sides.

18. Draw a regular hexagon with each side 5 cm. long.

19. Construct a regular hexagon on a given base.

20. Show how to construct a parallelogram given a diagonal, the angle which this diagonal makes with one of the sides, and the perimeter.

21. Through a given point P within an angle BAC construct a line which shall have its ends on AB and AC and be bisected at P. (*Hint.* Produce AP to Q so that PQ = AP.)

22. Draw a △ ABC with an obtuse angle at B. *Without producing* AB, construct through C a straight line perpendicular to AB.

23. If A and B are given points on opposite sides of a given straight line, find a point P on this line such that $\widehat{\mathsf{APB}}$ is bisected by the given line.

24. Construct a rhombus having two of its sides along two of the sides of a given triangle and having a vertex on the third side of the triangle.

TEST PAPERS

Paper I

1. A ship sets out from port, sails 10 miles due north, and then 7 miles due west. By means of a scale drawing find the final distance of the ship from the port.

2. Through how many degrees does (*a*) the seconds hand, (*b*) the minutes hand, (*c*) the hour hand, of a clock turn during five minutes?

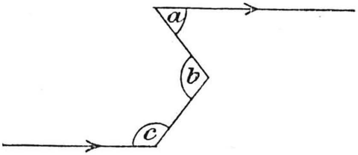


Fig. 206

3. Find a relation between *a*, *b*, and *c* in Fig. 206.

Paper II

1. The points B and C on a straight road are 400 yds. apart. A is a point 300 yds. from B and 350 yds. from C. Use a scale drawing to find the distance of A from the nearest point on the road.

2. In the triangle ABC, $\widehat{ABC}$ is 90° and BD is the perpendicular from B on to AC. Prove that $\widehat{BAD} = \widehat{DBC}$.

3. In Fig. 207 AB is parallel to CD. Find x.

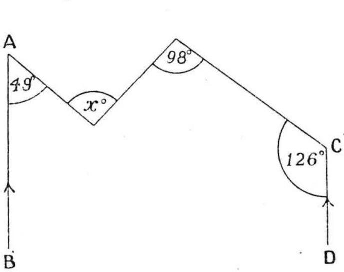


Fig. 207

Paper III

1. Show that in Fig. 208 $a + b = 180° + x$.

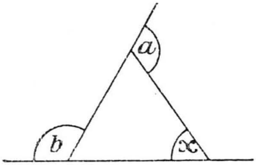


Fig. 208

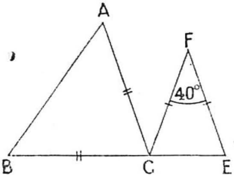


Fig. 209

2. Given that in Fig. 209 FE is parallel to AC, AC = CB, CF = FE, and $\widehat{CFE} = 40°$, calculate $\widehat{BAC}$.

3. In the quadrilateral ABCD, AD = DC, AC = CB, $\widehat{ADC} = 100°$, and $\widehat{BAD} = 90°$. Calculate $\widehat{DCB}$.

Paper IV

1. A town B is 17 miles west of a town A, and a third town C is 13 miles N. 18° W. of A. Find by drawing the distance from B to C.

2. Calculate x in Fig. 210.

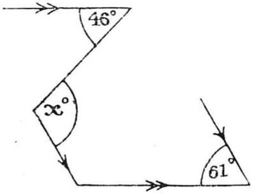


Fig. 210

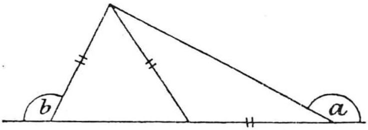


Fig. 211

3. Prove in Fig. 211 that $2a = 180° + b$.

Paper V

1. Three towns A, B, and C are situated so that AB = 7 miles, BC = 8·4 miles, and CA = 9·6 miles. If C is due north of A, and B is to the west of the line AC, find the bearing of B from C.

2. Calculate the value of angle x in Fig. 212.

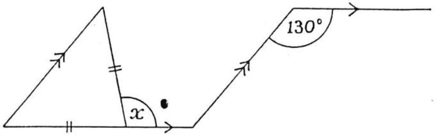


FIG. 212

3. The sides DA and CB of a quadrilateral ABCD are produced to X and Y. Prove that $\widehat{XAB} + \widehat{YBA} = \widehat{XDC} + \widehat{YCD}$.

PAPER VI

1. Given that M and N are the mid-points of the equal sides AC and AB of an isosceles △ ABC and MB meets NC at O, prove that △ OBC is isosceles.

2. In the quadrilateral ABCD, AB is parallel to DC, AC bisects $\widehat{BAD}$, $\widehat{CDA} = 80°$, and $\widehat{ABC} = 63°$. Which is greater, AB or AC?

3. Consider the following two theorems:

Theorem A. If, in the quadrilateral ABCD, AB is parallel to DC and AD is parallel to BC, then AD = BC.

Theorem B. If, in the quadrilateral ABCD, AB is parallel to DC and AD = BC, then AD is parallel to BC.

Theorem A is a true theorem.

Theorem B is a converse of Theorem A. Draw a figure which shows clearly that Theorem B is *not* true.

PAPER VII

1. The four angles of a quadrilateral are $x°$, $(x + 10)°$, $(x + 20)°$, and $(x + 30)°$. Find x.

2. The side BC of the △ ABC is produced to D. The bisectors of $\widehat{ABC}$ and $\widehat{ACD}$ meet at X. Given that $\widehat{BAC} = 70°$ and $\widehat{ABC} = 44°$, find $\widehat{BXC}$.

3. Given that in Fig. 213 AD = BC and $\widehat{ADC} = \widehat{BCD}$, prove that AC = BD.

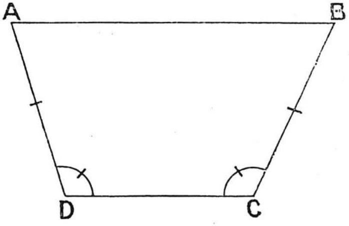


FIG. 213

PAPER VIII

1. Draw the $\triangle$ ABC in which AB = 3 in., $\widehat{ABC} = 44°$, and $\widehat{ACB} = 63°$. Measure AC.

2. Given that the angles of a quadrilateral, in order, are $2x°$, $3x°$, $4x°$, and $5x°$, prove that two of its sides are parallel.

3. Given that ABCD is a rectangle and P is a point such that PA = PB, prove that PC = PD.

PAPER IX

1. A vertical post AE stands in one corner of a rectangular playground ABCD. AB = 60 ft., BC = 70 ft., and AE = 40 ft. Find by drawing the distance of E from C.

2. In the $\triangle$ ABC, $\widehat{BAC} = 70°$, BD bisects $\widehat{ABC}$ and meets AC at D, and AB = BD. Which is greater, AC or BC?

3. In the $\triangle$ ABC, AB = BC and $\widehat{ABC} = 90°$. Any straight line is drawn outside the triangle through B, and perpendiculars AM and CN are drawn to this line. Prove that AM = BN.

PAPER X

1. Draw a quadrilateral ABCD with AB = 7 cm., BC = 7 cm., DC = 8 cm., DA = 6 cm., and $\widehat{DAB} = 60°$. Measure AC.

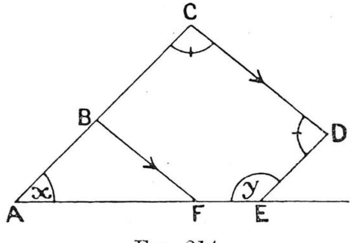


FIG. 214

2. In Fig. 214 AB = BF, BF $\parallel$ CD, and $\widehat{ACD} = \widehat{CDE}$. Prove that $y = 3x$.

3. Given that the diagonal of a quadrilateral bisects the angles through which it passes, prove that it bisects the other diagonal at right angles.

Paper XI

1. Draw a quadrilateral ABCD with AB = 3 in., $\widehat{DAB} = 80°$, AD = 2 in., DC ∥ AB, and BC = 2·4 in. Measure AC. How many solutions are there?

2. In the quadrilateral ABCD, AB = BD = DC, $\widehat{BAD} = 61°$, and $\widehat{BCD} = 58°$. Which is greater, AD or BC?

3. Prove that the three lines which bisect the sides of a triangle at right angles meet in a point. (*Hint.* Draw the perpendicular bisectors of AB and AC of △ ABC to meet at O. Join O to M, the mid-point of BC. Prove that OM is perpendicular to BC.)

Paper XII

1. In crawling once round an upright cylindrical tin 8 in. in circumference and 6 in. high a fly moves from a point on the upper edge to a point directly below on the lower edge. What is the length of the shortest path along which the fly could crawl?

2. A point X is taken inside the △ ABC such that BX > CX. Given that AB = AC, prove that $\widehat{ABX} > \widehat{ACX}$.

3. Prove that the three bisectors of the angles of a triangle meet in a point. (*Hint.* Let ABC be the triangle, and let the bisectors of $\widehat{BAC}$ and $\widehat{CBA}$ meet at I. Prove that IC bisects $\widehat{ACB}$.)

Paper XIII

1. Draw a parallelogram with sides 4 in. and 5 in. long and one diagonal 3·5 in. in length.

2. The side DA of a parallelogram ABCD is produced to X so that $\widehat{ABX} = 50°$, and AD is produced to M so that $\widehat{MCD} = 43°$. If AX = AB, which is greater, AX or DM?

3. In the square ABCD, E bisects AD and F bisects DC. Prove that AF is perpendicular to BE.

Paper XIV

1. Draw a △ ABC, given AB = 7 cm., AC = 9 cm., and the perpendicular from A to BC is 5 cm. long.

2. The side BC of the △ ABC is produced to D. The bisectors of $\widehat{ABC}$ and $\widehat{ACD}$ meet at X. If AC = CB prove that CB = CX.

3. In the square ABCD, X is any point on AD, and Y is a point on DC such that AY is perpendicular to BX. Prove that AY = BX.

PAPER XV

1. Draw a parallelogram with diagonals 6 cm. and 9 cm. long and the angle between the diagonals 73°. Measure the shorter side.

2. Given that in Fig. 215 AB = AC and BD is perpendicular to AC, prove that $y = 2\ \widehat{DBC}$.

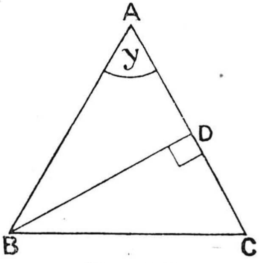


FIG. 215

3. Prove that the following construction for bisecting a given angle ABC is correct:

(i) Draw a circle with centre B and any radius to cut BA and BC at P and Q respectively.
(ii) Draw a circle with centre B and any other radius to cut BA and BC at X and Y respectively.
(iii) Join XQ and YP, and let K be the point of intersection.

Then BK bisects $\widehat{ABC}$.

PAPER XVI

1. Draw a parallelogram with diagonals of 6 cm. and 10 cm. and one side 5·5 cm. long. Measure the other side.

2. Calculate the interior angle of a regular ten-sided figure.

3. In the △s ABC, XYZ, AB = XY, $\widehat{ACB} = \widehat{XZY}$, and BC = YZ. Given that the triangles are not congruent, prove that

$$\widehat{BAC} + \widehat{YXZ} = 180°.$$

(*Hint.* Along ZX mark off ZP = CA, and prove that △ PYZ ≡ △ ABC.)

PAPER XVII

1. Draw a rhombus with diagonals 4 in. and 3 in. long.

2. If the perpendiculars from two vertices of a triangle on to the opposite sides are equal the triangle is isosceles. Prove this.

3. Given that ABCDEFG is a regular seven-sided polygon, calculate the angles of △ BGE.

PAPER XVIII

1. Draw the pentagon ABCDE, given that its angles are equal to one another, AB = BC = 1·5 in., and CD = 2 in. Measure AE.

2. In the quadrilateral ABCD, AD = CB and $\widehat{DAB} = \widehat{ABC}$. Prove that AB || DC.

3. In the parallelogram ABCD, AC = AD. Prove that BD is greater than AB and also greater than AD.

Paper XIX

1. Draw the figure ABCDEF in which AD = 3 in., the sides AB, BC, and CD are part of a regular hexagon, and AF, FE, and ED are part of a regular pentagon. Measure BE.

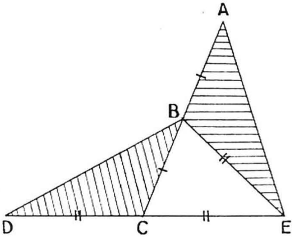


Fig. 216

2. In Fig. 216 AB = BC, and BE = EC = CD. Prove that △ ABE ≡ △ DCB.

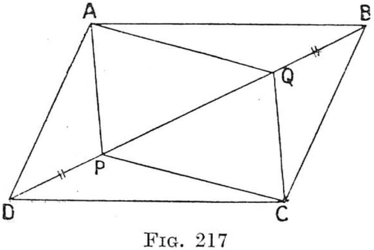


Fig. 217

3. Given that ABCD in Fig. 217 is a parallelogram and DP = BQ, prove that AQCP is a parallelogram.

Paper XX

1. On a given line as side construct a square using only a straight-edge and compasses.

2. ABCD in Fig. 218 is a parallelogram, and ABPQ and CBXY are squares. Prove that PX = BD.

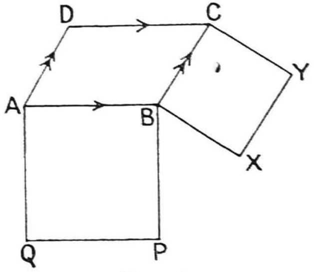


Fig. 218

3. Prove that the bisectors of the angles of a parallelogram with unequal sides form a parallelogram.

Paper XXI

1. Draw a straight line 4 in. long. Divide it into five equal parts, using only a straight-edge, set-square, and compasses.

2. Inside a △ ABC a point P is taken on the perpendicular from C to AB such that $\widehat{BPC}$ is the supplement of $\widehat{ABC}$. Perpendiculars PY and PX are drawn to AC and BC respectively. Given that AB = AC, prove that $\widehat{BPX} = \widehat{CPY}$.

3. Given that in Fig. 219 AB = AC and BL = CM, prove that LO = OM. (*Hint.* Draw LK parallel to AM to meet BC at K.)

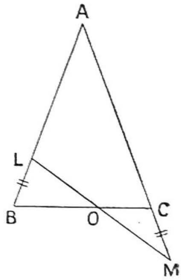


Fig. 219

Paper XXII

1. Construct a △ ABC on a given line BC as base such that AB = AC and $\widehat{BAC}$ = 1 right angle. (Use a straight-edge and compasses only.)

2. The perpendiculars from B and C meet the opposite sides of a △ ABC at Y and Z and intersect at O. Given that BO = CO, prove that ZY is parallel to BC.

3. In the △ ABC, AC is greater than AB, AM is a median, and the bisector of $\widehat{BAC}$ meets BC at N. Prove that

(*a*) CN > BN. (*b*) AM > AN.

PAPER XXIII

1. Show how to construct a triangle given the mid-points of its sides.

2. If every pair of alternate sides of an n-sided polygon be produced to meet the sum of the angles so formed is $(2n - 8)$ right angles. Prove this proposition.

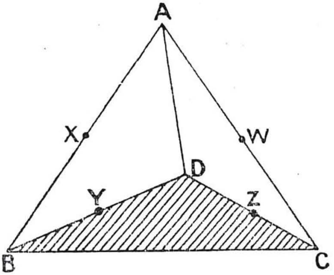


FIG. 220

3. Given in Fig. 220 that X, Y, Z, and W are the mid-points of the edges AB, BD, DC, and AC of the tetrahedron ABDC, prove that XYZW is a parallelogram. (*Hint.* Consider △ ABD; X and Y are the mid-points of its sides.)

PAPER XXIV

1. Draw the △ ABC, given the median AD = 3 in., $\widehat{BAD}$ = 40°, and $\widehat{CAD}$ = 60°. (*Hint.* Suppose AD produced to E, where DE = AD. What is the size of $\widehat{BEA}$? Now draw △ ABE.)

2. In the △ ABC, D is the mid-point of BC, E is the mid-point of AD, and BE is produced to cut AC at F. Prove that CF = 2AF.

3. Given that ABCD is a parallelogram and perpendiculars AX, BY, CZ, and DW are drawn to a given straight line outside the parallelogram, prove that AX + CZ = BY + DW.

Paper XXV

1. Draw any straight line AB. On AB construct a $\triangle$ ABC with $\widehat{ACB} = 120°$ and AC = BC without using a protractor.

2. X is a point inside a $\triangle$ ABC. Prove that

$$AB + AC > XB + XC.$$

3. In the $\triangle$s ABC and XYZ, AB = XY, BC = YZ, and the median BD = median YW. Prove that $\triangle ABC \equiv \triangle XYZ$. (*Hint.* Produce BD to E, making DE = BD, etc.)

Paper XXVI

1. A room is 9 ft. high, 11 ft. wide, and 12 ft. long. A fly crawls from one corner on the floor to the opposite corner on the ceiling, keeping to the walls. Find the length of its shortest path.

2. In the $\triangle$ ABC, AB = AC. Through a point D on BC a straight line is drawn cutting AC at F and AB produced at E. Given that FD = DE, prove that AF + AE = 2AB. (Compare Paper XXI, No. 3.)

3. If the diagonals of a quadrilateral are equal the lines joining the mid-points of the opposite sides of the quadrilateral are perpendicular. Prove this proposition.

Paper XXVII

1. In crawling twice round an upright cylindrical tin 8 in. in circumference and 6 in. high a fly moves from a point on the upper edge of the tin to a point directly below on the lower edge. What is the length of the shortest path along which the fly could crawl?

2. Equilateral triangles EAB, FBC, GCD, and HDA are described outside a parallelogram ABCD. Prove that EFGH is a parallelogram.

3. Prove that the sum of the lengths of two medians of a triangle is greater than the length of the third median. (See p. 114, Exercise 18.)

Paper XXVIII

1. Draw a straight line XY. Take two points A and B on the same side of XY. Find a point P on XY such that $\widehat{APX} = \widehat{BPY}$.

2. Given that P is any point within the $\triangle$ ABC, prove that

$$AB + BC + CA > AP + PB + PC > \tfrac{1}{2}(AB + BC + CA).$$

3. In the regular hexagon ABCDEF, BF meets AC at X, and BD meets AC at Y. Prove that AX = XY = YC.

Paper XXIX

1. Construct a triangle given the lengths of its medians.

2. Prove that a convex polygon cannot have more than three acute angles.

3. If two medians of a triangle are equal the triangle is isosceles. Prove this proposition.

Paper XXX

1. Given that, in $\triangle$s ABC and XYZ, AB = XY, AC = XZ, and $\widehat{BAC} > \widehat{YXZ}$, prove that BC > YZ.

2. Equilateral $\triangle$s ABX and ACY are described outside the $\triangle$ ABC. Given that AC > AB, prove that $\widehat{YBC} > \widehat{XCB}$. (*Hint.* Use the converse of the previous exercise.)

3. OLP, OMR, are straight lines such that OL = OM, OP = OR, and 3 $\widehat{POR}$ + 4 $\widehat{OPM}$ = 4 right angles. X is the intersection of LR and MP. Prove that OL = OX. (*Hint.* Use Paper XV, No. 3.)

PART III
AREAS

Definition of Altitude

ABC in Fig. 221 is a triangle with AD drawn perpendicular to BC (produced if necessary) to meet BC at D. If BC be taken as the base of the triangle, then the *altitude* of the triangle is defined to be AD.

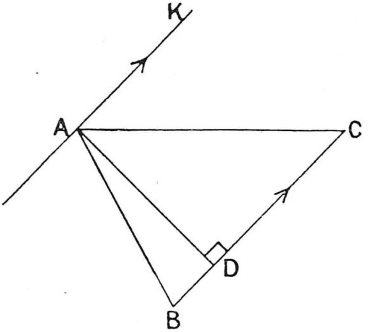


Fig. 221

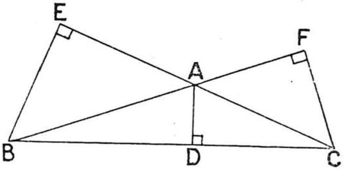


Fig. 222

Thus, in △ ABC of Fig. 222,

(i) if BC is the base, then AD is the altitude,
(ii) ,, CA ,, ,, ,, ,, BE ,, ,, ,,
and (iii) ,, AB ,, ,, ,, ,, CF ,, ,, ,,

If in Fig. 221 AK is drawn parallel to BC, then the perpendicular distance between AK and BC is equal to the length of AD (see p. 154).

If two triangles with equal altitudes lie on the same side of a common base line, then we may prove that this base line is parallel to the line joining their vertices. The triangles are then said to **lie between the same parallels.**

Distance between Two Parallel Lines

If AB and CD are two parallel lines and perpendiculars be drawn from any two points on AB, say X and P, to meet

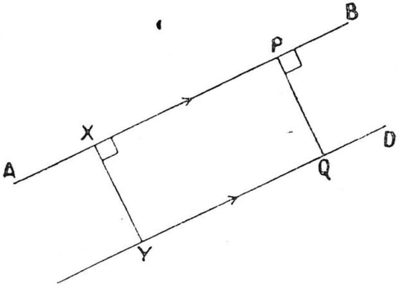


Fig. 223

CD at Y and Q (Fig. 223), then we can show that

(i) XY and PQ are both perpendicular to CD.

(ii) XY = PQ.

The length of PQ is said to be *the distance between* AB and CD.

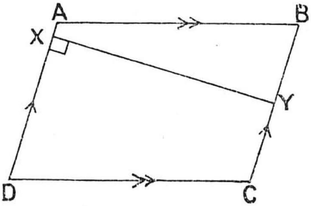


Fig. 224

If in a parallelogram ABCD (Fig. 224)

(i) AD be taken as the base, then the *altitude* is defined to be the *perpendicular* distance between AD and BC—namely XY.

(ii) AB is the base, then the altitude is the perpendicular distance between AB and CD.

NOTES ON THE AREA OF A RECTANGLE

If the lengths of the sides of a rectangle are x units and y units the area is xy square units.

(i) *x and y Whole Numbers.* The rectangle can be divided into xy squares with the side of each square equal to 1 unit.

$$\therefore \quad \text{area of rectangle} = xy \text{ sq. units.}$$

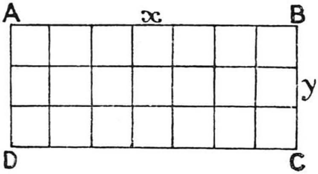


FIG. 225

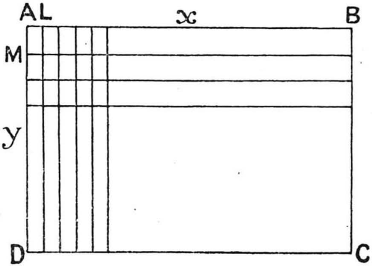


FIG. 226

(ii) *x and y Fractions.* Let $x = \dfrac{m}{n}$ and $y = \dfrac{p}{q}$, where m, n, p, and q are whole numbers.

Divide AB into m equal parts. Then n of these parts = 1 unit.
,, AD ,, p ,, ,, ,, q ,, ,, ,, = 1 unit.

Then, area of rectangle = area of mp small rectangles.
Also, area of nq small rectangles = 1 sq. unit.

$$\therefore \quad \text{area of rectangle} = \frac{mp}{nq} \text{ sq. units}$$
$$= \frac{m}{n} \times \frac{p}{q} \text{ sq. units}$$
$$= xy \text{ sq. units.*}$$

* We have assumed that x can be expressed in the form $\dfrac{m}{n}$, where m and n are whole numbers. In this case x is defined to be a *rational* or *commensurable* number. If x cannot be expressed in this form it is said to be *incommensurable*.

If either x or y is incommensurable the argument needs revision. The proof appropriate for this case will not be given in this book.

THEOREM 22

If a parallelogram and a rectangle have the same base and lie between the same parallels they are equal in area.

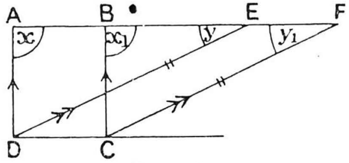


FIG. 227

Given the rectangle ABCD and the parallelogram CDEF on the same base DC and between the same parallels, DC and AF.

Prove that rect. ABCD = par^m CDEF in area.

Proof. In the $\triangle$s AED, BFC,

$$\begin{cases} x = x_1 & \text{(corr. angles; AD} \parallel \text{BC),} \\ y = y_1 & \text{(corr. angles; DE} \parallel \text{CF),} \\ \text{DE} = \text{CF} & \text{(opp. sides of a par}^{m}\text{).} \end{cases}$$

$\therefore$ $\triangle$ AED $\equiv$ $\triangle$ BFC (two angles, corr. side),
and, in particular,

$\triangle$ AED = $\triangle$ BFC in area.

Subtracting each triangle in turn from the quadrilateral AFCD, we have the remainders equal in area,

i.e., rect. ABCD = par^m CDEF in area. *Q.E.D.*

NOTE. No change occurs in the above proof if we substitute throughout the word 'parallelogram' for the word 'rectangle.' So exactly similar steps will prove:

Parallelograms on the same base and between the same parallels are equal in area.

The pupil should draw for himself the figure in which E lies betweeen A and B and convince himself that the proof still holds. In this case he should guard against saying, "Add to EBCD the $\triangle$s AED and BFC in turn, etc."

This would be a correct proof for the figure he had drawn, but would clearly require amending for the figure shown in the text, whereas the instruction "Subtract each triangle in turn from quadrilateral AFCD" applies to both figures.

Also if the base of the parallelogram be b units long and the altitude h units long, we know that the area of the rectangle is bh square units (Fig. 228).

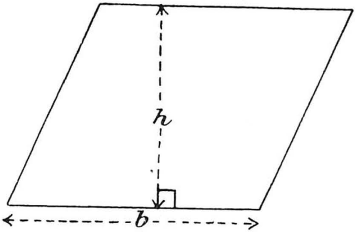


FIG. 228

$\therefore$ *area of parallelogram = bh sq. units.*

This algebraic expression for the area is often useful in riders. From it we deduce the following corollaries:

1. If two parallelograms have equal bases and equal altitudes their areas are equal.

Denote the length of the base by b units and that of the altitude by h units.

Then area of each par^{m} $= bh$ sq. units.

$\therefore$ the parallelograms are equal in area. *Q.E.D.*

2. If two parallelograms are equal in area and have equal bases their altitudes are equal.

The proof is left to the pupil.

3. If two parallelograms are equal in area and have equal altitudes their bases are equal.

The proof is left to the pupil.

These three results will be quoted when required.

THEOREM 23

The area of a triangle is equal to half the area of a rectangle drawn on the same base and between the same parallels.

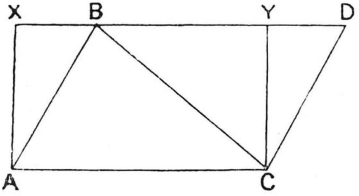


FIG. 229

Given a $\triangle$ ABC and a rectangle ACYX on the same side of a common base AC and between the same parallels AC and XY.

Prove that $\triangle$ ABC $= \frac{1}{2}$ rect. ACYX in area.

Construction. Draw CD $\parallel$ AB.

Proof. Since ABDC is a parallelogram (construction),

$\therefore$ $\triangle$ ABC $= \frac{1}{2}$ par^{m} ABDC in area (diagonal theorem).

But par^{m} ABDC $=$ rect. ACYX in area (base AC; $\parallel$s AC, XD).

$\therefore$ $\triangle$ ABC $= \frac{1}{2}$ rect. ACYX in area. *Q.E.D.*

NOTES. By virtue of Theorem 22 we may replace the word 'rectangle' by the word 'parallelogram.' Thus:

The area of a triangle is equal to half the area of a parallelogram drawn on the same base and between the same parallels.

If the length of AC is b units and the altitude of the triangle is h units, then

the area of triangle ABC *equals* $\frac{1}{2}bh$ *square units.*

From which, as in the previous theorem, we deduce the following corollaries:

1. Triangles with equal bases and equal altitudes have equal areas.

2. Triangles with equal bases and equal areas have equal altitudes.

3. Triangles with equal altitudes and equal areas have equal bases.

These results will be quoted when required.

For brevity we sometimes refer to the parallelogram ABCD (Fig. 230) as par^{m} AC, and to the rectangle ABCD as rect. AC.

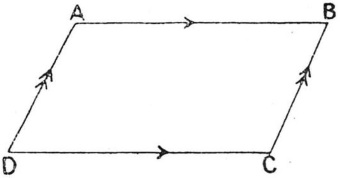


FIG. 230

Care should be taken to avoid ambiguity when two parallelograms have a common diagonal.

To speak of two parallelograms as 'equal' is meaningless unless we state the particular property of the parallelograms to which the equality refers—*e.g.*, we might say that they are equal in all respects, or equal in perimeter, or equal in area, and so on.

In practice, when the particular property is not stated *equality of area is implied.*

Thus,

par^{m} ABCD = rect. XYZW

should really be written

par^{m} ABCD = rect. XYZW in area.

When two figures are equal in area they are sometimes said to be *equivalent* figures.

Area of a Trapezium

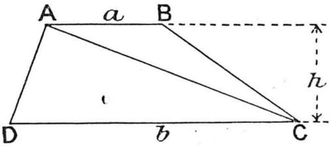


Fig. 231

If in Fig. 231 the parallel sides AB and DC are a in. and b in. long respectively and the perpendicular distance between them is h in. we have

Area of $\triangle$ ADC $= \frac{1}{2}bh$ sq. in.

Area of $\triangle$ ABC $= \frac{1}{2}ah$ sq. in.

$\therefore$ area of trapezium ABCD $= (\frac{1}{2}bh + \frac{1}{2}ah)$ sq. in.,

i.e., area of trapezium ABCD $= \frac{1}{2}h\ (a + b)$ sq. in.

Area of a Quadrilateral

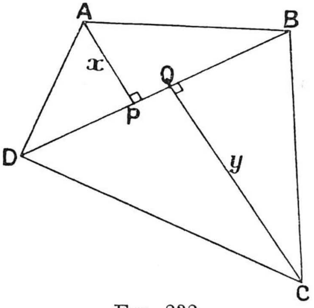


Fig. 232

In Fig. 232 AP $= x$ in., CQ $= y$ in., and BD $= a$ in.

Area of $\triangle$ ABD $= \frac{1}{2}ax$ sq. in.

Area of $\triangle$ CBD $= \frac{1}{2}ay$ sq. in.

$\therefore$ area of quad[l] ABCD $= (\frac{1}{2}ax + \frac{1}{2}ay)$ sq. in.,

i.e., area of quad[l] ABCD $= \frac{1}{2}a\ (x + y)$ sq. in.

BC = 3 in., $\widehat{DAB}$ = 100°, and $\widehat{ABC}$ = 110°. Find the area of the quadrilateral, making any necessary measurements.

18. On graph paper plot the points (3, 1), (8, 3), (9, 6), (5, 7), and (1, 4). Calculate the area of the figure formed by joining these points in order.

19. A point O is taken inside the △ ABC so as to be equidistant from the three sides. Given that the area of △ ABC is 5 sq. in. and the perimeter of △ ABC is $12\frac{1}{2}$ in., calculate the distance of O from any side.

20. The area of a △ ABC is 10 sq. in., and L is the mid-point of the median AD. Calculate the area of △ ABL.

21. In a given parallelogram ABCD, L is the mid-point of AB and M is a point on AD such that AM = 2MD. Prove that the area of △ ALM equals one-sixth the area of parallelogram ABCD.

22. E is any point on the side BC of a parallelogram ABCD, and DE and AB are produced to meet at F. Prove that △ ABE = △ CEF in area.

23. Show that the converse of Exercise 8 is not necessarily true. (*Hint.* Show that P may lie on the straight line drawn through A parallel to BC.)

24. In the quadrilateral ABCD, K is any point on AB. Through A and B parallels are drawn to KD and KC respectively to meet DC (produced) at L and M. Prove that △ KLM equals quadrilateral ABCD in area.

25. In △ ABC, D is a point on BC such that CD equals 2DB, and E is a point on AD such that AD equals 4AE. What fraction is the area of △ AEB of the area of △ ABC?

26. Points E and D are taken on the sides AC and AB of a △ ABC such that △ ABE equals △ ADC in area. Prove that DE is parallel to BC.

27. Given that L and M are the mid-points of the sides AC and AB of a △ ABC, prove, *without quoting* that LM is parallel to CB, that △ BLC equals △ BMC in area. *Hence prove* that LM is parallel to CB.

28. Any point K is taken inside the parallelogram ABCD. Prove that △ ABK + △ CDK = △ ADK + △ BCK in area.

29. If BD bisects the quadrilateral ABCD, BD also bisects AC. Prove this proposition.

30. Prove that the area of a rhombus is half the area of the rectangle contained by its diagonals.

31. Any point L is taken on the side BC of a parallelogram ABCD. A parallelogram ALMN is drawn so that MN passes

through D. Given that △ ABL = △ ADN in area, prove that MC is parallel to DL.

32. Given that in Fig. 237 AB ∥ CD ∥ EF and BD = DF, prove that △ ADE = △ BCF in area.

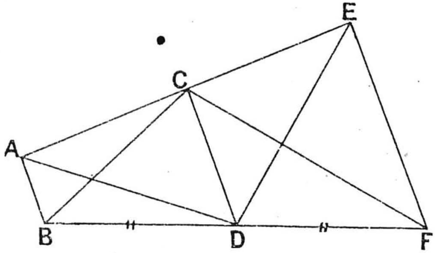


FIG. 237

33. Prove that △ ADE and △ BCF of Fig. 237 each equals half the area of the quadrilateral AEFB.

34. Two triangles have a pair of sides of the one respectively equal to a pair of sides of the other, and the angles contained by these sides are supplementary. Prove that the triangles are equal in area.

35. A line is drawn parallel to the diagonal DB of the parallelogram ABCD to cut the sides BC and DC at X and Y respectively. Prove that △ ABX = △ ADY in area.

36. The diagonals of a given parallelogram ABCD meet at O, and L is any point on DC (not produced). Prove that

(*a*) Area of △ ADL + area of △ BLC = constant.
(*b*) Area of △ OAL + area of △ BOL = constant.

37. The medians AX and BY of a △ ABC meet at G. Prove that quad^l GXCY = $\frac{1}{3}$ △ ABC in area.

38. In the parallelogram ABCD, X is the mid-point of DC and Y is a point on AB such that AY equals 2YB. What fraction of the parallelogram ABCD is the trapezium AXCY?

39. In the △s ABC and DEF, AB = DE, BC = EF, $\widehat{ABC} = 90°$, and $\widehat{DEF}$ is acute. Prove that △ DEF is less than △ ABC in area. If $\widehat{DEF}$ is obtuse, which triangle has the greater area?

40. Given two intersecting straight lines OALM, OBXY, P a point within the angle MOY, OA = LM, and OB = XY, prove that △ PLM + △ PXY = quad^l OAPB in area. (Compare Exercise 24 on p. 192.)

LINKS

To C 120 To D 140	To A 300 250 180 100 From B	90 to E

41. Find the area in square yards of the field ACDBE (Fig. 238) from the above table, the meaning of which is: BN = 100 links,

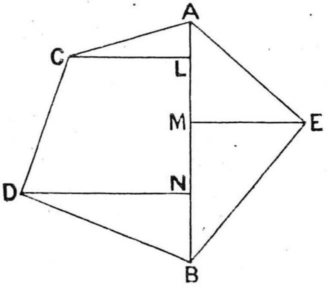


FIG. 238

BM = 180 links, BL = 250 links, BA = 300 links, LC = 120 links, ND = 140 links, and ME = 90 links. (100 links = 1 chain = 22 yds.)

42. Calculate the area in square yards of the field whose dimensions are given in the following table:

LINKS

To F 100	To A 410 370 300 190 120 From B	150 to C 110 to D 80 to E

Theorem 24: Pythagoras' Theorem

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.

First Method

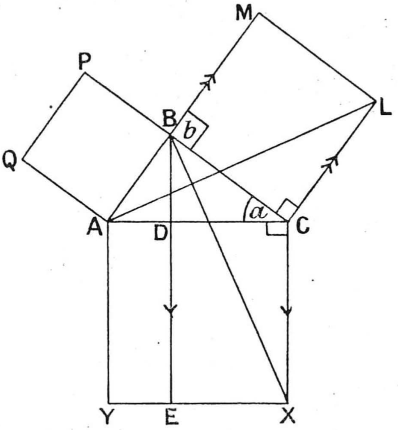


Fig. 239

Given a △ ABC with $\widehat{ABC}$ a right angle, and squares AX, BL, and BQ drawn on the sides of the triangle.

Prove that sq. AX = sq. BL + sq. BQ in area.

Construction. Draw BDE ∥ CX. Join AL, BX.

Proof. We know that

$$\widehat{ABC} = 1 \text{ right angle} \quad \text{(given)},$$

$$\text{and} \quad b = 1 \text{ right angle} \quad \text{(angle in a square)}.$$

$$\therefore \quad \widehat{ABM} = 2 \text{ right angles}.$$

$\therefore$ ABM is a straight line.

In the $\triangle$s BCX, LCA,

$$\begin{cases} CX = CA & \text{(sides of square)}, \\ CB = CL & \text{(sides of square)}, \\ \widehat{XCB} = \widehat{ACL} & (\text{each} = 90^\circ + a). \end{cases}$$

$\therefore$ $\triangle$ BCX $\equiv$ $\triangle$ LCA (two sides, included angle),

and, in particular,

$\triangle$ BCX = $\triangle$ LCA in area.

Also $\triangle$ BCX = $\frac{1}{2}$ rect. CE in area (base CX; ||s CX, BE)

and $\triangle$ ACL = $\frac{1}{2}$ sq. BL in area (base CL; ||s CL, AM).

$\therefore$ rect. CE = sq. BL in area.

Similarly, we may prove

rect. AE = sq. BQ in area.

Adding, we get

sq. AX = sq. BL + sq. BQ in area. *Q.E.D.*

NOTE. Referring to Fig. 239, we may note that the area of *the rectangle* DCXE *is sometimes written* DC.CX and the area of *the square* BCLM *is sometimes written* BC^2.

With this notation we might write the statement 'rectangle AE = square BQ' in the form 'AY.YE = AB^2.'

Second Method

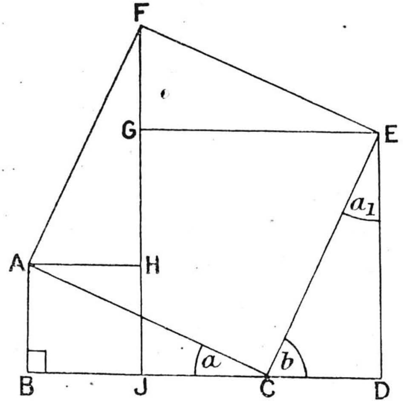


FIG. 240

Given a $\triangle$ ABC with $\widehat{ABC}$ a right angle.

Prove that $AC^2 = AB^2 + BC^2$.

Construction. Draw the square AFEC.
Draw FJ and ED perpendicular to BC.
Draw AH and EG perpendicular to FJ.

Proof. We have $a + b = 90°$ $(\widehat{ACE} = 90°)$
and $a_1 + b = 90°$ $(\widehat{D} = 90°)$.

$$\therefore \quad a = a_1.$$

In the $\triangle$s ABC, CDE,

$$\begin{cases} AC = CE & \text{(sides of square),} \\ a = a_1 & \text{(proved),} \\ \widehat{B} = \widehat{D} & \text{(rt. angles).} \end{cases}$$

$\therefore$ $\triangle ABC \equiv \triangle CDE$ (two angles, corr. side),

and, in particular,

$$BC = DE,$$

and area of $\triangle$ ABC = area of $\triangle$ CDE.

$= X$, say.

Similarly, we may prove that

area of $\triangle$ FGE $= X$,

and area of $\triangle$ AHF $= X$.

Now the area of ABDEF may be split up in two ways:

(i) Sq. AE + $\triangle$ ABC + $\triangle$ CDE.

(ii) Sq. AJ + sq. DG + $\triangle$ AHF + $\triangle$ FGE.

Therefore we have

$$AC^2 + 2X = AB^2 + DE^2 + 2X.$$

$$\therefore \quad AC^2 = AB^2 + DE^2,$$

$$i.e., \quad AC^2 = AB^2 + BC^2 \quad (BC = DE).$$

Q.E.D.

THEOREM 25: CONVERSE OF PYTHAGORAS' THEOREM

If the square on one side of a triangle is equal to the sum of the squares on the other two sides the triangle is right-angled.

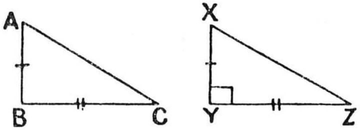


FIG. 241

Given a $\triangle$ ABC in which $AC^2 = AB^2 + BC^2$.

Prove that $\widehat{B}$ = a right angle.

Construction. Draw a $\triangle$ XYZ in which

$$XY = AB,$$
$$\widehat{Y} = 90°,$$
$$YZ = BC.$$

Proof. Since $AC^2 = AB^2 + BC^2$ (given),

$\therefore\ AC^2 = XY^2 + YZ^2$ $(AB = XY,\ BC = YZ)$.

Also $XZ^2 = XY^2 + YZ^2$ (Pythagoras' Theorem).

$\therefore\ AC^2 = XZ.^2$

$\therefore\ AC = XZ.$

In the $\triangle$s ABC, XYZ,

$$\begin{cases} AB = XY & \text{(construction)}, \\ BC = YZ & \text{(construction)}, \\ CA = ZX & \text{(proved)}. \end{cases}$$

$\therefore\ \triangle ABC \equiv \triangle XYZ$ (three sides),

and, in particular,

$$\widehat{B} = \widehat{Y}.$$

But $\widehat{Y} = 90°$ (construction).

$\therefore\ \widehat{B}$ = a right angle. *Q.E.D.*

EXERCISES : XXXIII

Section A (Rational Numbers Only)

1. The △ ABC has a right angle at C.

(*a*) If CB = 3 in. and CA = 4 in., what is the length of AB?
(*b*) If CB = 5 in. and AB = 13 in., what is the length of CA?
(*c*) If AB = 25 in. and AC = 24 in., what is the length of BC?

2. The side of a square is 6 in. long. What is the area of the square on its diagonal?

3. Find the area of a rectangle whose diagonal is 10 in. long and one of whose sides is 8 in. long.

4. Calculate the altitude of an isosceles triangle whose base is 8 in. long and whose equal sides are each 5 in. long.

5. The sides of a triangle are 13 in., 13 in., and 10 in. long. Find:

(*a*) The altitude drawn on to the short side.
(*b*) The area of the triangle.
(*c*) The altitude drawn on to a long side.

6. Find AD in Fig. 239 when AB = 3 in. and BC = 4 in.

7. Find BC in Fig. 239 when AC = 10 cm. and DC = 2·5 cm.

8. Find DC in Fig. 239 when AB = 6 in. and AC = 9 in.

9. Two vertical posts 4 ft. apart have their tops 10 ft. and 7 ft. above the ground. Find the distance between their tops.

10. If the sides of a triangle are x, y, and z units long, in which of the following cases is the triangle right-angled:

(*a*) $x = 3,\ y = 4,\ z = 5.$
(*b*) $x = 4,\ y = 5,\ z = 6.$
(*c*) $x = 11,\ y = 61,\ z = 60.$
(*d*) $x = 25,\ y = 65,\ z = 60.$
(*e*) $x = l,\ y = l + m,\ z = m.$
(*f*) $x = l^2 + m^2,\ y = 2lm,\ z = l^2 - m^2.$

11. In the △ ABC, AD is an altitude. Given that AD = 6 cm., BD = 4 cm., and DC = 9 cm., prove that $\widehat{BAC} = 90°$.

12. One angle of a right-angled triangle is 60° and the hypotenuse is 2 in. long. Show that the area of the square on the side opposite the angle of 60° is 3 sq. in.

13. In a triangle whose sides are a, b, and c units long $b^2 = 3a^2$ and $c^2 = 4a^2$. Show that the angles of the triangle are 90°, 60°, and 30°.

14. The diagonals of the quadrilateral ABCD are perpendicular. Prove that $AB^2 + CD^2 = BC^2 + DA^2$.

15. In the $\triangle$ ABC, AD is an altitude. Prove that

$$AC^2 - AB^2 = DC^2 - DB^2.$$

16. Prove that in any rhombus the sum of the squares on the sides is equal to the sum of the squares on the diagonals.

17. Prove that in a right-angled isosceles triangle the area of the square on the hypotenuse is equal to four times the area of the triangle.

18. X and Y are any points on the sides BA and AC of a $\triangle$ ABC in which $\widehat{BAC} = 90°$. Prove that

$$BY^2 + CX^2 = XY^2 + BC^2.$$

19. Prove that if AB and BC are sides of a regular hexagon

$$AC^2 = 3AB^2.$$

20. In the right-angled $\triangle$ ABC, AB = AC, and the bisector of $\widehat{ABC}$ meets AC at D. Prove that $CD^2 = 2AD^2$. (*Hint.* Draw a perpendicular from D on to BC.)

Section B (includes Irrational Numbers)

21. Calculate the length of a side of a rhombus whose diagonals are 10 cm. and 12 cm. long.

22. Find the slant height of a cone whose height is 8 in. and the radius of whose base is 3 in.

23. Calculate x in Fig. 242.

24. Find the distance between the points (2, 3) and (4, 7).

25. Prove that the triangle formed by the points (1, 1), (4, 7), and (6, 6) is right-angled.

26. Is the triangle formed by the points (2, 8), (4, 7), and (6, 11) right-angled?

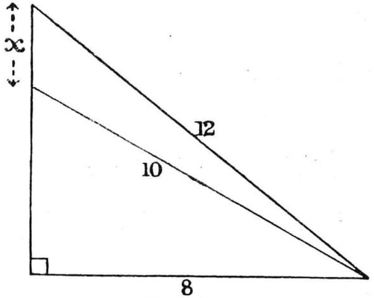


FIG. 242

27. In the quadrilateral ABCD, $\widehat{DAB} = 90°$, AD = 17 cm., AB = 31 cm., DC = 5 cm., and CB = 35 cm. Prove that $\widehat{BCD} = 90°$.

28. In the $\triangle$ ABC, AB = AC, the altitude BE = 6 in., and EC = 3 in. Calculate the area of the triangle.

29. Calculate the length of the diagonal of a cube whose edge is 8 cm. long.

30. Calculate the length of the diagonal of a rectangular box whose edges are 4 in., 7 in., and 5 in. long.

31. A pyramid, whose triangular sides are isosceles, is constructed with a square base with sides 4 in. in length. The height of the pyramid is 6 in. Find:

(*a*) The length of the slant edge.
(*b*) The total surface area.

32. A and B are opposite corners of a box with edges 4 in., 5 in., and 6 in. long. Find the length of the shortest route from A to B (*a*) through space, (*b*) along the sides of the box.

33. Prove that if P is any point outside a rectangle ABCD

$$PA^2 + PC^2 = PB^2 + PD^2.$$

What happens if P is inside the rectangle?

34. In the $\triangle$ ABC, BE is an altitude and AB = AC. Prove that $BC^2 = 2AB.CE$.

35. Given that AD is an altitude of the equilateral $\triangle$ ABC and $\triangle$ ADE is also equilateral, prove that the perimeter of $\triangle$ABC is equal to four times an altitude of $\triangle$ ADE.

36. The $\triangle$ ABC has a right angle at B. Points P and Q are taken on BC so that BP = PQ = QC. Prove that

$$8AQ^2 = 3AC^2 + 5AP^2.$$

37. In Fig. 239 BA is produced to T so that TA = AB. Prove that CT = QY.

38. From the result of the previous exercise prove that in Fig. 239 $XL^2 + YQ^2 = 5AC^2$.

39. Prove that in Fig. 239 BY is perpendicular to CQ.

40. Use the preceding exercise to give another proof of Exercise 38.

41. How many right-angled triangles can be constructed each with its perimeter less than 10 in. and having two of its sides an integral number of inches?

42. Prove that if a square and a rectangle have the same perimeter the square has the greater area.

43. Points E, F, are taken on the sides BC, CD, of a square ABCD. Given that AB = a in., BE = x in., DF = y in., prove that if $ya + ax = x^2 + a^2$, then $\widehat{AEF}$ is a right angle.

CONSTRUCTIONS

CONSTRUCTION 9

To construct a triangle equal in area to a given quadrilateral.

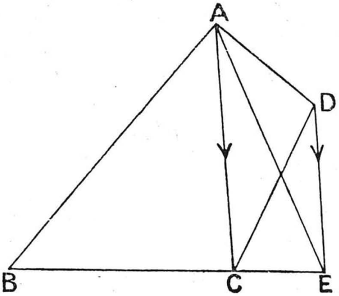


FIG. 243

Given a quadrilateral ABCD.

Construct a triangle with its area equal to the area of quadrilateral ABCD.

Construction. Draw a diagonal AC.

Through D draw DE parallel to AC, meeting BC produced at E.

Join AE.

Then $\triangle$ ABE = quad[l] ABCD in area.

Proof. We know that

$\triangle$ ACD = $\triangle$ ACE in area (base AC; ||s AC, DE).

To each triangle add $\triangle$ ABC.

$\therefore$ quad[l] ABCD = $\triangle$ ABE in area.

$\therefore$ The $\triangle$ ABE has been constructed equal in area to the quad[l] ABCD.

Corollary. To construct a triangle having the same area as a given polygon, say ABCDEF . . .

Fig. 244 shows the polygon AQDEF . . . with the same area as the polygon ABCDEF . . .

Also the new polygon has one side less than the original polygon.

Reducing in this way, step by step, we arrive at a triangle having the same area as the original polygon.

If the polygon had the shape given in Fig. 245, we should have to vary our work thus:

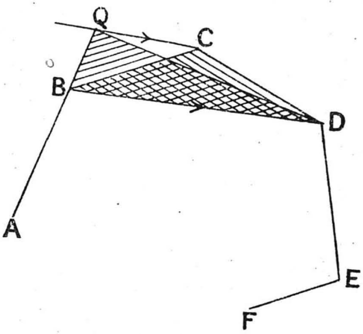


Fig. 244

By drawing a line CQ through C parallel to DB we have

$$\text{fig. ABCDEF} \ldots = \text{fig. ABDEF} \ldots - \triangle \text{BCD}$$
$$= \text{fig. ABDEF} \ldots - \triangle \text{BQD}$$
$$= \text{fig. AQDEF} \ldots$$

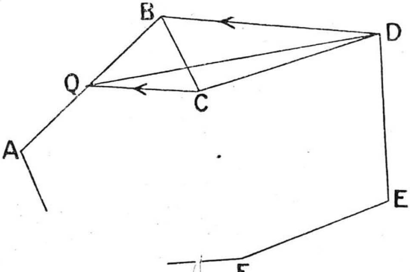


Fig. 245

The polygon ABCDEF . . . has thus been reduced to another polygon AQDEF . . . having one side less than the original.

By successive steps the original polygon can be reduced to a triangle.

EXAMPLE. Show how to construct through a given point on the base of a triangle a line to bisect the triangle.

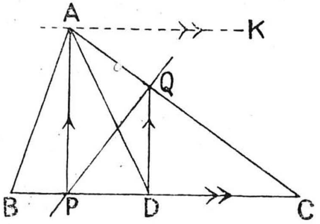


FIG. 246

Given a △ ABC with P a point on BC.

Construct a line through P to bisect the triangle.

Construction. Bisect BC at D.
Join AD and AP.
Draw QD parallel to AP.
Then PQ is the required line.

Proof. We know that

△ QDP = △ QDA in area (base QD; ||s QD, AP).

Adding to each △ QDC, we have

△ QPC = △ ADC in area.

Also △ ADC = △ ABD in area (base BD = DC; ||s BC, AK).

∴ △ ADC = $\frac{1}{2}$ △ ABC in area.

∴ △ QPC = $\frac{1}{2}$ △ ABC.

∴ PQ bisects △ ABC.

How must the above construction be modified if P lies between D and C?

EXAMPLE. To show how to construct a triangle with a given base and base angle whose area shall be equal to that of a given triangle.

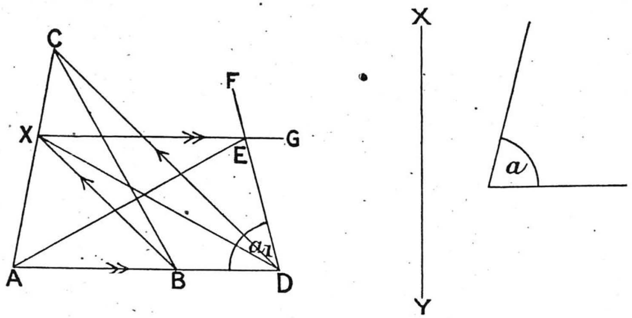


FIG. 247

Given a $\triangle$ ABC, a straight line XY, and an angle a.

Construct a triangle with the base equal to XY, a base angle equal to a, and the area equal to the area of $\triangle$ ABC.

Construction. (i) Along AB mark AD = XY.
(ii) Join DC.
(iii) Draw BX $\parallel$ DC.
(iv) Draw XG $\parallel$ AD.
(v) Make $\widehat{ADF} = a$.
(vi) Join AE.

Then AED is the required triangle.

Proof. We know that

$\triangle$ XBC = $\triangle$ XBD in area (base XB; $\parallel$s XB, CD).

Adding $\triangle$ ABX to each, we have

$\triangle$ ABC = $\triangle$ ADX in area.

Also $\triangle$ AED = $\triangle$ AXD in area (base AD; $\parallel$s AD, XG).

$\therefore$ $\triangle$ AED = $\triangle$ ABC in area.

Also AD = XY (construction),

and $a_1 = a$ (construction).

$\therefore$ $\triangle$ AED is the required triangle.

Exercises: XXXIV

1. Draw a △ ABC of area 3 sq. in. in which AB = 3 in. and $\widehat{ABC} = 50°$.

2. Draw a parallelogram with sides 5 cm. and 6 cm. in length and with an area of 23 sq. cm.

3. Draw a rhombus of side 2·5 in. and area 4 sq. in.

4. Draw a triangle with area 4 sq. in., base of length 2·5 in., and median bisecting the base 4·3 in. in length.

5. Draw any parallelogram. Construct a rhombus of equal area, having a side of the parallelogram as base.

6. Bisect a given triangle by a line drawn through a vertex.

7. On the diagonal of a given square as base construct an isosceles triangle equal in area to the square.

8. Construct a right-angled isosceles triangle equal in area to a given square.

9. Construct four lines through the vertex of a given triangle to divide the triangle into five equal parts.

10. On the diagonal of a given square as base construct a rectangle equal in area to the square.

11. Through a vertex of a given parallelogram construct two lines to trisect the parallelogram.

12. Draw the convex quadrilateral ABCD with AB = 5 cm., $\widehat{CAB} = 25°$, AD = 6 cm., $\widehat{CBD} = 40°$, and BD = 4·5 cm. Without further measurement, construct a △ BAE having the same area as the quadrilateral.

13. In the quadrilateral of the previous exercise take a point P on BA such that BP = 1 in. Construct a △ PLM equal in area to the quadrilateral LM lying along DC, produced if necessary.

14. Draw a △ ABC with BC = 8 cm., AB = 7 cm., and CA = 6 cm. On the same side of BC as A find a point D such that BD = 4·5 cm. and CD = 5·5 cm. Construct a triangle with A as one vertex equal in area to the re-entrant quadrilateral ABDC.

15. Draw a regular hexagon ABCDEF. Construct a rectangle on AC as base having the same area.

16. Draw a regular pentagon ABCDE. Construct a triangle of equal area with base along AE and vertex C.

17. Construct a rectangle equal in area to a given quadrilateral.

18. Draw a triangle ABC with AB = 4 in., AC = 3·5 in., and BC = 3 in. Produce BC to D so that CD = 1·2 in. Without further measurement, find a point P on AB such that

$$\triangle \text{BPD} = \triangle \text{ABC in area.}$$

19. With A, B, C, and D as in the preceding exercise, construct a rhombus with BD as base equal in area to △ ABC.

20. On a given line as base construct a rectangle equal in area to a given rectangle.

21. Draw a △ ABC with AB = 7 cm., $\widehat{ABC} = 40°$, and $\widehat{BCA} = 65°$. On AB make AD = 3 cm. Bisect the △ ABC by a line through D.

22. With A, B, C, and D as in the previous exercise, trisect the △ ABC by two lines through D.

23. Construct a parallelogram equal in both area and perimeter to a given triangle.

24. On a given straight line as base construct a rhombus equal in area to a given rectangle. If the sides of the given rectangle are 2 in. and 8 in. in length, what is the least length for the given straight line that will give a solution?

25. Bisect a given quadrilateral by a line through an angular point.

26. On a given side of a given quadrilateral construct a rectangle equal in area to the quadrilateral.

27. Draw any square with side x in. Construct a line whose length is (*a*) $\sqrt{2}\,x$ in., (*b*) $\sqrt{7}\,x$ in., (*c*) $\sqrt{37}\,x$ in., (*d*) $\frac{\sqrt{3}}{4}\,x$ in.

28. Construct a square equal in area to the sum of two given squares.

29. Construct a square equal in area to the difference between two given squares.

30. Construct a square equal in area to the sum of three given squares.

LOCUS

Revise p. 49.

If we take a fixed point O and allow the point P to move *on the plane of the paper* so that the distance OP is always 4 cm., P must move on the circumference of the circle centre O and radius 4 cm. (Fig. 248). Further, if we take any point

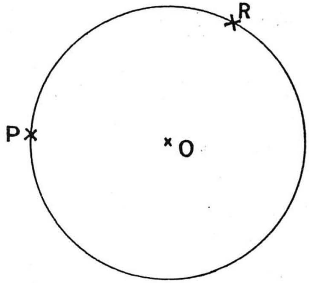


FIG. 248

R on the circumference of this circle the distance OR equals 4 cm. In this case the circle is said to be *the locus* of the point P which *moves* so as to satisfy the condition that the distance of P from O is 4 cm.

This suggests the following formal *definition*, applicable only to figures in *one plane*, of the word *locus*.

A line is (or lines are) said to be the *locus* of the moving point if

(i) when the point satisfies a given condition (or conditions) it is on the line (or lines), *and*

(ii) when the point is on the line (or lines) it satisfies the given condition (or conditions).

We may extend this definition to apply to figures in space by replacing the word *line* in the above definition by the word *surface*.

THEOREM 26 (i)

A point which is equidistant from two given points lies on the perpendicular bisector of the line joining the given points.

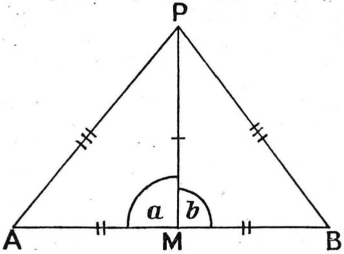


FIG. 249

Given two points A and B and a point P such that PA = PB.

Prove that P lies on the perpendicular bisector of AB.

Construction. Join P to M, the mid-point of AB.

Proof. In the $\triangle$s APM, BPM,

$$\begin{cases} AP = BP & \text{(given)}, \\ PM = PM, & \\ AM = BM & \text{(construction)}. \end{cases}$$

$\therefore \quad \triangle APM \equiv \triangle BPM$ (three sides),

and, in particular,

$$a = b.$$

Also $\quad a + b = 180°$ (half a revolution).

$$\therefore \quad a = 90°.$$

Hence PM is perpendicular to AB;

i.e., P lies on the right bisector of AB. *Q.E.D.*

THEOREM 26 (ii)

Any point on the perpendicular bisector of a straight line joining two given points is equidistant from the given points.

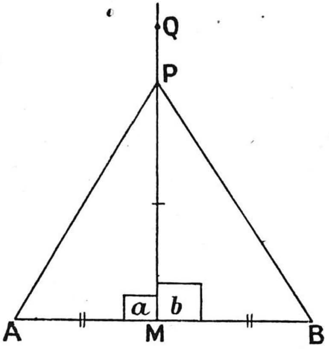


FIG. 250

Given two points A and B with QM the perpendicular bisector of AB and P any point on MQ.

Prove that PA = PB.

Proof. In the $\triangle$s PAM, PBM,

$$\begin{cases} AM = BM & \text{(given)}, \\ PM = PM, & \\ a = b & \text{(each a right angle, given).} \end{cases}$$

$\therefore$ $\triangle$ PAM $\equiv$ $\triangle$ PBM (two sides, included angle),

and, in particular,

PA = PB. *Q.E.D.*

FIRST LOCUS THEOREM. From Theorems 26 (i) and (ii) follows:

THEOREM 26 (iii)

The locus of a point which moves so as to be equidistant from two given points is the perpendicular bisector of the line joining the given points.

THEOREM 27 (i)

A point which moves within an angle formed by two intersecting straight lines and is equidistant from the lines moves on the bisector of the angle.

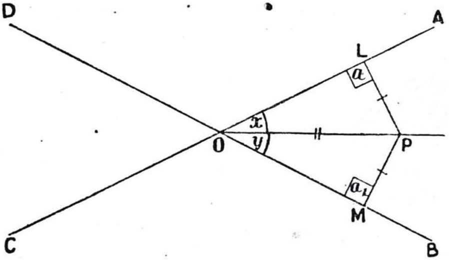


FIG. 251

Given two fixed intersecting straight lines AOC and BOD and P any point which moves within the angle AOB so that PL = PM, where PL and PM are perpendicular to OA and OB respectively.

Prove that $x = y$.

Proof. In the $\triangle$s LOP, MOP,

$$\begin{cases} PL = PM & \text{(given)}, \\ PO = PO, & \\ a = a_1 & \text{(each a right angle)}. \end{cases}$$

$\therefore$ $\triangle$ LOP $\equiv$ $\triangle$ MOP (two sides, non-included rt. angle), and, in particular,

$x = y$. *Q.E.D.*

THEOREM 27 (ii)

Any point on the bisector of a given angle is equidistant from the arms of the angle.

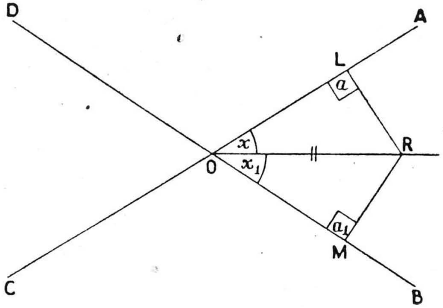


FIG. 252

Given two fixed lines AOC and BOD and R any point within $\widehat{AOB}$ and on the bisector of $\widehat{AOB}$.

RL and RM are perpendicular to OA and OB respectively.

Prove that RL = RM.

Proof. In the $\triangle$s RLO, RMO,

$$\begin{cases} x = x_1 & \text{(given),} \\ a = a_1 & \text{(each a right angle),} \\ \text{OR} = \text{OR}. \end{cases}$$

$\therefore$ $\triangle$ RLO $\equiv$ RMO (two angles, corresponding side),

and, in particular,

RL = RM. *Q.E.D.*

SECOND LOCUS THEOREM

THEOREM 27 (iii)

The locus of points which are equidistant from two intersecting straight lines is the pair of straight lines which bisect the angles between the given lines.

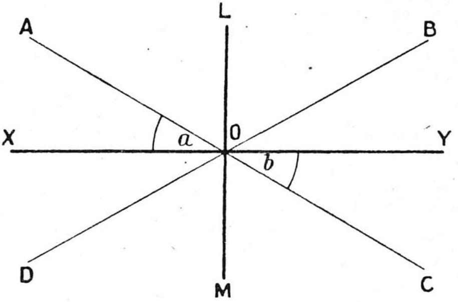


FIG. 253

For by Theorems 27 (i) and (ii), if the points which are equidistant from AC and BD lie within $\widehat{BOC}$, their locus is the line OY which bisects $\widehat{BOC}$.

Similarly, if the points lie within $\widehat{AOD}$, their locus is the line OX which bisects $\widehat{AOD}$.

Now, $a = b$ (halves of vert. opp. angles)

and $a + \widehat{XOC} = 180°$ (AOC a st. line).

$\therefore$ $b + \widehat{XOC} = 180°$.

$\therefore$ XOY is a straight line.

Similarly if OL, OM, bisect $\widehat{AOB}$, $\widehat{COD}$, respectively, then LOM is a straight line, and the complete locus of points which are equidistant from AC and BD is the pair of lines XY and LM. *Q.E.D.*

EXAMPLE. Given a straight line AB with P a fixed point on it (Fig. 254), what is the locus of a point which moves so as to satisfy the following two conditions:

(i) Its distance from AB is 2 cm.

and (ii) Its distance from P is not more than 4 cm.

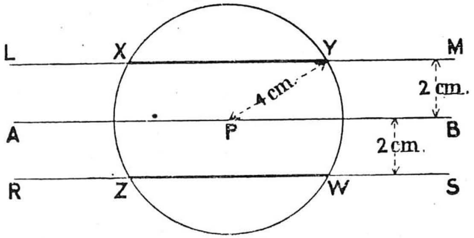


FIG. 254

Clearly the point must be (i) on LM or RS, (ii) within the circle, centre P and radius 4 cm.

Further, every point on the terminated lines XY and ZW is (i) 2 cm. from AB, (ii) not greater than 4 cm. from P.

Hence the required locus consists of the two terminated straight lines XY and ZW.

If in the above example we remove the restriction that the point is to move in one plane, its locus is clearly the curved surface of the cylinder obtained by rotating the terminated line XY about the line AB.

EXAMPLE. To find the locus of a point which moves so that the difference between its distances from two given intersecting straight lines is constant.

Given two fixed straight lines AOC and BOD and a variable point P such that the difference between the distances of P from AC and BD is constant, equal to x, say (Fig. 255).

Find the locus of P.

Work. Consider P within $\widehat{AOB}$ and such that PY $-$ PX $= x$, where PY is perpendicular to DB and PX is perpendicular to AC.

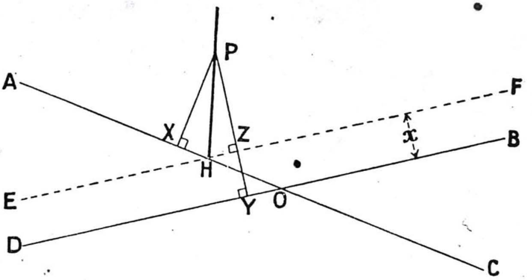


FIG. 255

Draw EF ∥ DB and distance x from DB, on the same side of DB as P. Let PY meet EF at Z.

$$\begin{aligned} \text{Since}\quad PY - PX &= x \quad \text{(given)} \\ \text{and}\quad PY - PZ &= x \quad \text{(construction)}, \\ \therefore\quad PX &= PZ. \end{aligned}$$

Further, any point P such that PX = PZ satisfies the condition that PY − PX = x.

Therefore the locus of P is the bisector of $\widehat{AHF}$

(Second Locus Theorem).

Now consider P within $\widehat{AOB}$ and such that PX − PY = x (Fig. 256).

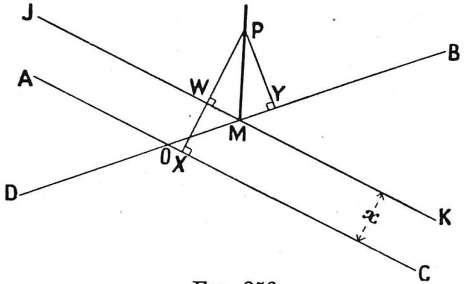


FIG. 256

On the same side of AC as P draw JK ∥ AC and distance x from AC.

As before, the locus of P is now the bisector of $\widehat{JMB}$.

Now remove the restriction that P is to lie within $\widehat{AOB}$, and we get for the complete locus of P the 'cross' shown in Fig. 257.

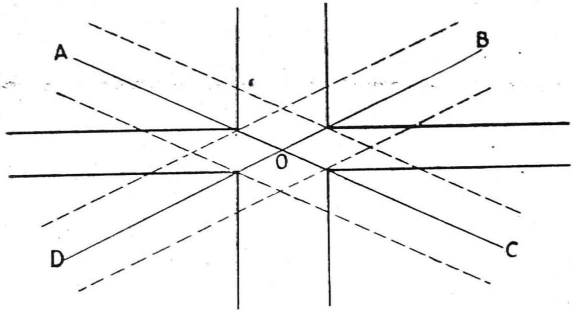


FIG. 257

The general method of solving a locus problem is as follows:

(i) By drawing or, better, by intuition, guess which line (or lines) forms the locus.

(ii) Prove that the moving point can have no position except on this line or these lines.

(iii) Prove that every point on this line (or these lines) satisfies the given condition (or conditions).

The method is illustrated in the following example.

EXAMPLE. A variable parallelogram ABCD has a fixed base AB and the vertices C and D move so that the diagonal BD remains of constant length (Fig. 258). Find the locus of C.

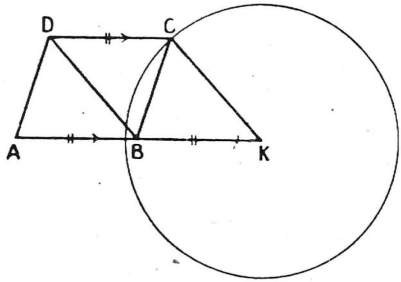


FIG. 258

Analysis. We notice that the locus of D is a circle centre B, radius BD.

While D and C move, the distance DC and the direction of DC remain unchanged. This suggests for the locus of C an equal circle with centre at K where $BK = AB$.

Given a parallelogram ABCD with AB fixed and BD fixed in length.

Find the locus of C.

Construction. Produce AB to K, making $BK = AB$.

Join KC.

Then the locus of C is the circle centre K and radius equal to BD.

Proof. (i) $DC = AB$ (opp. sides of par^{m}),

and $AB = BK$ (construction).

$\therefore$ $DC = BK$.

Also $DC \parallel BK$ ($DC \parallel AB$).

$\therefore$ $DB = CK$ (opp. sides of par^{m}).

Now DB is given of fixed length.

$\therefore$ CK is of fixed length.

Also K is a fixed point.

$\therefore$ C lies on a circle centre K and radius equal to BD.

(ii) Take any point C_1 on this circle.

Complete parallelogram BKC_1D_1.

As above, C_1D_1 is equal and parallel to AB.

$\therefore$ ABC_1D_1 is a parallelogram,

and $BD_1 = KC_1$ (opp. sides of par^{m} BKC_1D_1).

Also $KC_1 = BD$ (construction).

$\therefore$ $BD = BD_1$.

$\therefore$ the parallelogram ABC_1D_1 satisfies the given conditions.

Hence the locus of C is the circle centre K and radius equal to BD.

The next example illustrates how loci are used in constructing a figure from given measurements.

EXAMPLE. Draw a △ ABC of area 3 sq. in., with BC 3 in. long and the median through A 2·2 in. long.

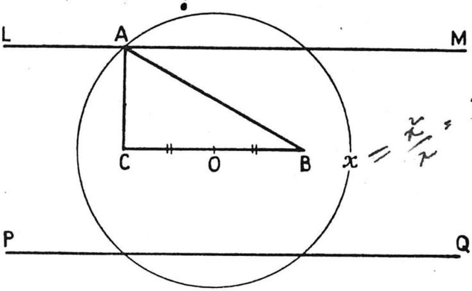


FIG. 259

Analysis. Since BC = 3 in. and the area of △ ABC = 3 sq. in., the perpendicular from A to BC equals 2 in.

The locus of points which are 2 in. from BC consists of the two straight lines LM and PQ parallel to BC.

Also if O is the mid-point of BC we are told that the median OA is 2·2 in. in length.

The locus of points which are 2·2 in. from O is the circle x, centre O and radius 2·2 in.

Hence A may have any one of the four positions in which the circle x meets LM and PQ.

Draw a △ ABC with BC 3 in. and median AO 2·2 in. long, and its area 3 sq. in.

Construction. (i) Draw two lines LM and PQ parallel to BC and 2 in. from it.

(ii) Draw a circle with radius 2·2 in. and centre O, where O is the mid-point of BC.

Let A be any one of the four points in which this circle cuts LM and PQ.

Then △ ABC is a triangle satisfying the given conditions.

As an exercise the pupil should prove that the four triangles so obtained are congruent.

Before the following exercises are attempted some of the locus exercises on pp. 49–51 should, of course, have been worked through.

EXERCISES: XXXV

For each of the first six questions draw a $\triangle$ ABC with AB = 6 cm., AC = 7 cm., and BC = 8 cm.

1. Find a point P within the $\triangle$ ABC which is 3 cm. from A and 2·4 cm. from BC.

2. Find a point P which is equidistant from B and C and is 4 cm. from A. How many solutions are there?

3. Find a point P which is equidistant from BA and CA and is 3 cm. from BC. How many solutions are there?

4. Find a point P which is equidistant from A and C and is equidistant from AC and BC. How many solutions are there?

5. Find a point P which is equidistant from A, B, and C.

6. Find a point P which is equidistant from AB, BC, and CA.

7. Prove that the locus of a variable point which is at a given distance from a fixed straight line consists of two straight lines parallel to the given line.

8. What is the locus of the vertices of isosceles triangles which have the same given base?

9. Find the locus of points which are equidistant from two given parallel straight lines.

10. What is the locus of the variable vertex of a triangle whose area is constant and whose base is fixed?

11. In the fixed $\triangle$ ABC, P is a variable point on BC (not produced) and Q bisects AP. Find the locus of Q.

12. The variable $\triangle$ ABC has a given area, and the base BC is fixed. Given that BD is a median, prove that the locus of D consists of two straight lines parallel to BC.

13. In the $\triangle$ ABC the point A is fixed and the length of BC is given. The area of $\triangle$ ABC is given, and D is the foot of the perpendicular from A to BC. What is the locus of D?

14. Find the locus of the variable point of intersection of the diagonals of a parallelogram whose base is fixed and whose area is given.

15. Given that A and B are fixed points and P moves so that $PA^2 - PB^2$ is given, prove that the locus of P is a straight line perpendicular to AB.

16. Use the first locus theorem (p. 182) to prove that **the three right bisectors of the sides of a triangle meet in a point.**

17. Use the second locus theorem (p. 185) to prove that **the three bisectors of the angles of a triangle meet in a point.**

18. The vertex P of a rectangle PQRS is fixed, and the sides PQ and PS lie along fixed lines. The diagonal QS is of given length. What is the locus of R?

19. The point P moves on a fixed circle, O is a fixed point outside the circle, and Q is the mid-point of OP. Find the locus of Q.

20. In the quadrilateral ABCD the side AB and the diagonal AC are fixed at right angles. What is the locus of D if

$$DA^2 + AB^2 = DC^2 + BC^2?$$

21. The medians of the $\triangle$ ABC meet at G. The base BC is fixed, and the area of $\triangle$ ABC is given. Prove that the locus of G consists of two straight lines parallel to BC.

22. Find the locus of a point which moves so that the sum of its distances from two given straight lines is constant.

23. Given a fixed $\triangle$ ABC and a point P which moves so that $\triangle$ PAB $=$ $\triangle$ PAC in area, prove that the locus of P consists of two straight lines. (*Hint.* Compare Exercise 23 on p. 163.)

24. Given two fixed straight lines LM, XY, and a variable point P such that the sum of the areas of $\triangle$s PLM, PXY is constant, find the locus of P. (*Hint.* Compare Exercise 40 on p. 164.)

TEST PAPERS

Paper XXXI

1. Draw a △ ABC of area 2·25 sq. in. with AB = 3 in. and $\widehat{BAC}$ = 60°. Measure CB.

2. The diagonals AC and BD of a parallelogram ABCD meet at E. AC is produced to M, making CM = EC, and BD is produced to N, making DN = ED. Prove that △ BMC = △ AND in area.

3. Calculate the length of CB in Exercise 1. Give your answer in inches, correct to two decimal places.

Paper XXXII

1. Draw a right-angled triangle in which the sides about the right angle are 6 cm. and 8 cm. in length. (*a*) Measure the third side. (*b*) Calculate the length of the third side.

2. The sides of a parallelogram are 8 cm. and 12 cm. long, and the longer sides are 5 cm. apart. Find the distance between the shorter sides.

3. Given that in Fig. 260 AF = FB and AE = EC, prove that △ BXF = △ CXE in area.

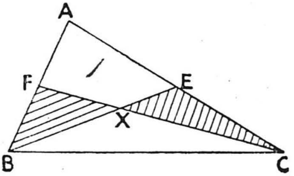


Fig. 260

Paper XXXIII

1. Copy Fig. 261 without using set-square or protractor.

(*a*) Measure AB.

(*b*) Calculate AB.

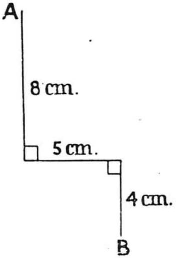


Fig. 261

2. In the △ ABC, $\widehat{ABC}$ = 90°. Through a point D on AB a line DE is drawn parallel to BC to cut AC at E. If AD = a in., AB = b in., DE = x in., and BC = y in., prove that $ay = bx$.

3. Prove that the sum of the squares on the four sides of a rhombus is equal to the sum of the squares on its diagonals.

Paper XXXIV

1. Draw a △ ABC of area 5 sq. in. in which AB = 4 in. and $\widehat{BAC}$ = 40°. Measure $\widehat{ABC}$.

2. ABCD and ABFE are two parallelograms on the same side of AB. Prove that:

(*a*) DCFE is a parallelogram.

(*b*) The area of the parallelogram ABCD is equal to the sum or difference of the areas of the parallelograms ABFE and DCFE.

3. In the quadrilateral ABCD, AB = 3 in., BC = 4 in., the area of △ ADC is 6 sq. in., and the perpendicular from D to AC is 2·4 in. long. Show that $\widehat{ABC}$ = 90°.

Paper XXXV

1. Draw a △ ABC with AB = 3 in., $\widehat{BAC}$ = 40°, and $\widehat{BCA}$ = 72°. Find all the points which are equidistant from the lines CA and CB and are also 3·5 in. from B.

2. Given that the quadrilaterals ABDC and ABEC are equal in area, prove that DE is parallel to BC.

3. Find, to the nearest inch, the length of the longest straight stick which can be packed in a rectangular box whose sides are 6 ft., 5 ft., and 3 ft. long.

Paper XXXVI

1. Draw a rectangle ABCD with AB 8 cm. and BC 5 cm. long. On AB as base draw a rhombus having the same area as the rectangle.

2. The heights of the tops of two vertical posts above level ground are 20 ft. and 25 ft. Find the distance between their tops when the posts are 12 ft. apart.

3. The quadrilateral ABCD has a given area, and the sides AB and AD are fixed in size and position. What is the locus of C?

Paper XXXVII

1. Draw to scale a rectangular garden 50 yds. wide and 80 yds. long. Draw the locus of points in the garden which are 10 yds. from the nearest edge of the garden, and also more than 20 yds. from each corner of the garden.

2. In the quadrilateral ABCD, $\widehat{ABC} = \widehat{ACD} = 90°$. The areas of the squares described on the sides AB, BC, and CD are respectively 5 sq. in., 7 sq. in., and 4 sq. in. Calculate AD.

3. In the $\triangle$ ABC, $\widehat{BAC}$ is greater than $\widehat{BCA}$, and P is any point on the bisector of $\widehat{ABC}$. Prove that the area of $\triangle$ PBC is greater than the area of $\triangle$ PBA.

Paper XXXVIII

1. Draw a regular pentagon with each side 2 in. long. Draw a triangle having the same area.

2. If in Fig. 233 (p. 161) the units are inches and the area of the parallelogram is A sq. inches, prove that

$$\frac{h+k}{x+y} = \frac{A}{xy}.$$

3. Points P and Q are taken on the sides AD and DC of a square ABCD. AQ and BP intersect at right angles at X. Prove that

$\triangle$ AXB = quad[l] PXQD in area.

Paper XXXIX

1. On squared paper draw the locus of points the sum of whose distances from two given intersecting perpendicular lines is 5 in.

2. In the $\triangle$ ABC, BA = BC and $\widehat{A} = 55°$. AB is produced to D so that $\widehat{ADC} = 30°$. Which has the greater area, $\triangle$ ABC or $\triangle$ BCD?

3. Given that in Fig. 262 $\triangle$s ACD and ABE are equal in area, prove that BD is parallel to CE.

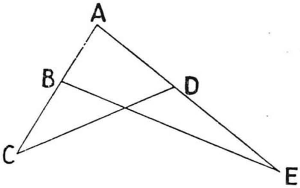


Fig. 262

Paper XL

1. Draw a parallelogram ABCD with AC 9 cm., BD 11 cm., and AB 7 cm. in length. On AC as base draw a rectangle having the same area as the parallelogram.

2. In the pentagon ABCDE, AB is fixed, E, D, and C vary so that AE = BC, $\widehat{DEA} = \widehat{BCD}$, and ED = CD. What is the locus of D?

3. In the $\triangle$ ABC, AD is an altitude. Given that AB > AC, prove that BD > DC.

Paper XLI

1. In the hexagon ABCDEF, AB || FC || ED. The distance between AB and FC is 3 in., the distance between FC and ED is 4 in., and AB is 5 in., FC is 7 in., and ED is 2 in. in length. Find the area of the hexagon.

2. In the quadrilateral ABCD, $\widehat{ABC}$ is 90°. Prove that when $AB^2 - AD^2 = CD^2 - CB^2$, then $\widehat{D} = 90°$.

3. In the △ ABC, of given area, BC is fixed and AB < BC. What is the locus of A?

Paper XLII

1. Mark a point on the edge of a large coin. Place the coin on a piece of paper and roll it (without slipping) along the edge of a ruler. The marked point will trace out a curve. By marking a number of positions of the point on your paper sketch the curve, which is called a cycloid.

2. One side of a parallelogram is 6·5 cm. long, and the diagonals are 12 cm. and 5 cm. long. Prove that the parallelogram is a rhombus.

3. Given that in Fig. 263 ABCD and APQR are parallelograms, P lies on DC, and B lies on QR, prove that the parallelograms are equal in area.

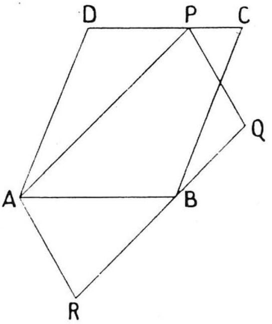


Fig. 263

Paper XLIII

1. Make a scale drawing of a field from the information given in the following table. Calculate the area of the field in acres.

Yards

	B	
	200	
40	150	
	100	80
	80	30
	A	

2. A point P is taken on the side BC of a $\triangle$ ABC. Given that AB = AC = a in., AP = b in., BP = c in., and CP = d in., prove that $a^2 - b^2 = cd$.

3. In the quadrilateral ABCD, $\widehat{ABC} = \widehat{BCD} = 90°$, and E is a point on BC. Prove that when AB = a in., BE = b in., EC = c in., and CD = d in., then the area of $\triangle$ AED = $\frac{1}{2}(ac + bd)$ sq. in.

Paper XLIV

1. Draw a triangle in which AB = 5 cm., BC = 6 cm., and CA = 7 cm. Find a point P satisfying the three conditions (a) PA = PC, (b) P is 3 cm. from AC, and (c) PB is greater than 4 cm.

2. The perimeter of a parallelogram ABCD is 30 in. and its area is 30 sq. in. The distance between AB and CD is 3 in. Find the distance between AD and BC.

3. If in Exercise 3 of Paper XLIII $ad = bc$ prove that

$$\widehat{AED} = 90°.$$

Paper XLV

1. Calculate the area of the convex pentagon ABCDE in which AB = BC = ED = 3 in., AE = 4 in., and $\widehat{EAB} = \widehat{BCD} = \widehat{CDE} = 90°$.

2. If two sides of a triangle are given in length the area of the triangle is greatest when the sides contain a right angle. Prove this proposition.

3. Draw any isosceles triangle ABC in which AB = AC and AB > BC. Produce BC to D, making BD = AB. Using a straight-edge and compasses only, construct an isosceles triangle (on BD as base) whose area is equal to the area of $\triangle$ ABC.

Paper XLVI

1. In Fig. 264 AC = CE = 13 cm., AB = 12 cm., and ED = 5 cm. Calculate BD.

2. Calculate the area of $\triangle$ ACE in Fig. 264.

3. In the $\triangle$ ABC of constant area the base BC is fixed and $\widehat{CAB}$ is variable. If BE is a median of the triangle, what is the locus of E?

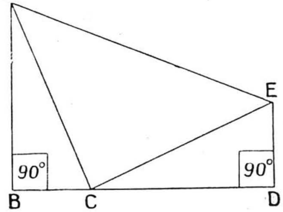


Fig. 264

Paper XLVII

1. Draw a $\triangle$ ABC with AB = 2 in., BC = 2·5 in., and CA = 3 in. Find a point P inside the triangle such that P is equidistant from AC and CB and is 1 in. from AB. Measure PA.

2. BE is a median of the $\triangle$ ABC, and EC is a median of the $\triangle$ EBD. Prove that $\triangle$ AEB = $\triangle$ ECD in area.

3. ABCD in Fig. 265 is a square. AL is any line inside the square, and CN is drawn parallel to AL. BM and DO are drawn perpendicular to AL. Prove that

(*a*) $\triangle$ ALB $\equiv$ $\triangle$ MBC.
(*b*) OLMN is a square.

Deduce Pythagoras' Theorem.

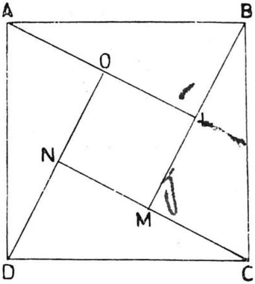


Fig. 265

Paper XLVIII

1. Reproduce Fig. 266 with the given measurements. Draw the locus of points which move inside the figure so as to satisfy these two conditions: (*a*) the points must be 1 in. from the nearest point on the boundary of the figure; (*b*) the points must be more than 2 in. from A, B, D, and C.

2. The medians BE and CF of a $\triangle$ ABC meet at G. Prove that

$\triangle$ BCG = quadrilateral AFGE in area.

3. A square board with sides 8 ft. long is supported in a horizontal position by four equal chains of length 6 ft. passing from its corners to a point above the board. Find the distance of this point from the board.

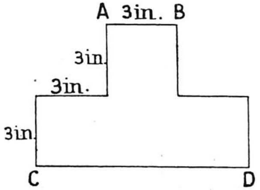


Fig. 266

Paper XLIX

1. Draw a quadrilateral ABCD with AD = DC = 2·5 in., AB = BC = 3 in., and $\widehat{DAB} = 100°$. Find a point P inside the quadrilateral such that P is equidistant from AD and DC and also such that PB = PC.

2. The base of a rectangular box is ABCD, and AE is an edge through A. Prove that $EB^2 + BC^2 = ED^2 + DC^2$.

3. A point X is taken inside the △ ABD of the parallelogram ABCD. Prove that

(*a*) △ BCX + △ ADX = ½ par^m ABCD in area.
(*b*) △ BXD = △ BCX − △ ABX in area.

Paper L

1. Draw a square ABCD with AB 6 cm. long. On AC as base draw an isosceles triangle equal in area to the square.

2. Given that, in △s ABC and XYZ, AB = XY, BC = YZ, and $\widehat{B} + \widehat{Y} = 180°$, prove that △ ABC = △ XYZ in area.

3. The base of a rectangular box is ABCD, and ALMB is one of its sides. An insect crawls from M to D, keeping to the side ALMB and to the base ABCD. If AB = 5 ft., BC = 8 ft., and BM = 4 ft., what is the length of its shortest path?

Paper LI

1. Draw a △ ABC of area 4 sq. in. in which AC is 3 in. long and the median CE is 2·8 in. long.

2. The diagonals AC and BD of a parallelogram meet at E, and X is any point on AB. Prove that

△ BCE − △ XCE = △ DEX in area.

3. The top of a folding card table is the square ABCD, with sides $2a$ in. in length. The leg BE can be folded so that E coincides with C. Given that the leg BE be turned into the position BX, and AX = b in., prove that $b^2 = 8a^2$.

Paper LII

1. Construct a quadrilateral ABCD of area $8\frac{1}{2}$ sq. in. with AB = 3 in., $\widehat{B} = 90°$, BC = 4 in., and AD = $1\frac{1}{2}$ in.

2. Three of the edges of a cube are AB, AC, and AD. The area of △ ABC is x sq. in., and the area of △ BCD is y sq. in. Prove that $y^2 = 3x^2$.

3. Prove that the area of the parallelogram formed by joining the mid-points of a given quadrilateral is equal to half the area of the given quadrilateral.

PAPER LIII

1. Draw a quadrilateral ABCD with AD = AB = 3 in., $\widehat{A} = \widehat{D} = 100°$, and $\widehat{B} = 70°$. Find a point X on CD produced such that △ BCX equals quadrilateral ABCD in area.

2. Given that in Fig. 267 rectangle AC equals rectangle EG in area, prove that CE is parallel to FB.

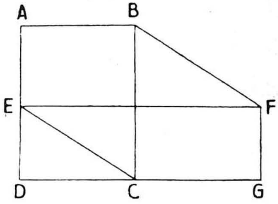


FIG. 267

3. In the △ ABC, $\widehat{BAC}$ is 90° and AD is an altitude. Given that AB = c in., CA = b in., and BD = p in., prove that

$$p\sqrt{c^2 + b^2} = c^2.$$

PAPER LIV

1. The area of the trapezium ABCD in Fig. 268 is 246 sq. in. Prove that $\widehat{DBC} = 90°$.

2. In the rectangle ABCD the sides AB and AD lie on fixed lines, and the *length* of BD is constant. What is the locus of C?

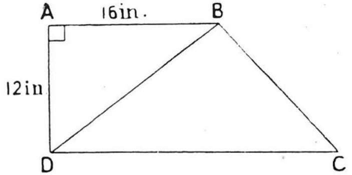


FIG. 268

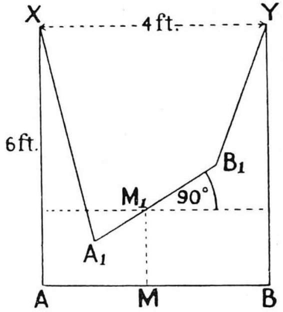


FIG. 269

3. A horizontal bar AB, length 4 ft., is suspended by two equal vertical strings XA and YB (see Fig. 269). The bar AB is twisted through 90° into the position A_1B_1 so that its centre M rises vertically to the position M_1 and the strings, XA_1 and YB_1, remain taut. Calculate the distance MM_1.

PART IV

THE CIRCLE

THE locus of a point which moves in a plane so that its distance from a fixed point in the plane is constant is the circumference of a circle. The figure so obtained is a circle, the boundary line is the circumference, and the fixed point is the centre.

We shall prove (see Theorem 30) that only one circle can be drawn to pass through three given points. *Hence no ambiguity arises when we speak of the circle* ABC.

If O is the centre of the circle ABC we call OA a *radius* of the circle. OB and OC are also radii (Fig. 270).

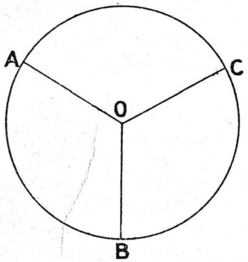


FIG. 270

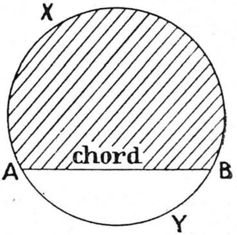


FIG. 271

Chord. The straight line joining two points on the circumference of a circle is called a *chord* of the circle (Fig. 271).

A chord of a circle which passes through the centre of the circle is called a *diameter* of the circle.

Concentric circles are circles which have the same centre.

CIRCUMSCRIBING CIRCLE OF A TRIANGLE

If a circle be drawn through the vertices of a triangle it is called the *circumscribing* circle or *circum-circle* of the triangle. The centre of the circle is the *circum-centre* of the triangle. The radius of the circle is the *circum-radius*. (See Construction 11.)

CHORDS OF A CIRCLE

Theorem 28

A straight line drawn from the centre of a circle to bisect a given chord which is not a diameter is at right angles to the chord.

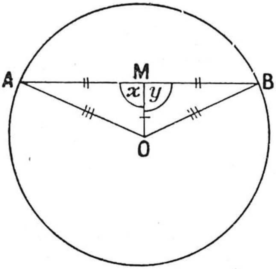


Fig. 272

Given AB a chord of a circle centre O, and AM = MB.

Prove that $x = 90°$.

Construction. Join OA, OB.

Proof. In the $\triangle$s AMO, BMO,

$$\begin{cases} AM = BM & \text{(given)}, \\ MO = MO, & \\ OA = OB & \text{(radii)}. \end{cases}$$

$$\therefore \triangle AMO \equiv \triangle BMO \text{ (three sides)},$$

and, in particular,

$$x = y.$$

Since $x + y = 180°$ (half a revolution),

$$\therefore 2x = 180°$$

$$\therefore x = 90°.$$

Q.E.D.

THEOREM 29

If a line be drawn from the centre of a circle perpendicular to a chord of the circle it will bisect the chord.

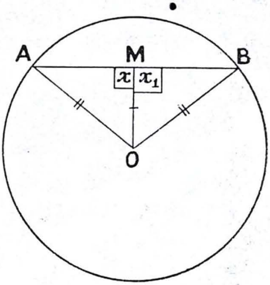


FIG. 273

Given AB a chord of a circle centre O, and OM a perpendicular to AB.

Prove that AM = MB.

Construction. Join OA, OB.

Proof. In the △s AOM, BOM,

$$\begin{cases} OA = OB & \text{(radii)}, \\ OM = OM, & \\ x = x_1 & \text{(given rt. angles)}. \end{cases}$$

$\therefore$ △ AOM ≡ △ BOM

(two sides, non-included rt. angle),

and, in particular,

AM = MB. *Q.E.D.*

THEOREM 30

There is one circle, and only one, which passes through three points not in the same straight line.

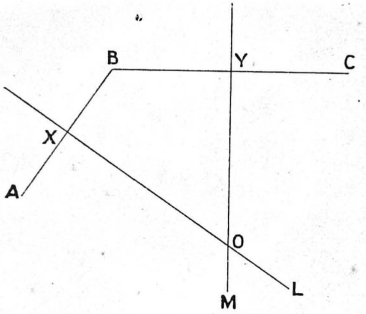


FIG. 274

Given three points A, B, and C not in the same straight line.

Prove that (i) a circle can be drawn through A, B, and C.
(ii) Only one such circle can be drawn.

Construction. Draw the perpendicular bisectors of AB and BC, say XL and YM.

Proof. (i) Since AB and BC are not in the same straight line,

∴ XL is not parallel to YM.

∴ XL meets YM, say at O.

Now OA = OB (OX the perp^{r} bisector of AB),

and OB = OC (OY the perp^{r} bisector of BC).

∴ OA = OB = OC.

Thus, with centre O and radius OA we can draw a circle to pass through A, B, and C. *Q.E.D.*

(ii) We know that the centre of any circle which passes through A and B must lie on XL (First Locus Theorem).

And the centre of any circle through B and C lies on YM.

Hence the centre of any circle which passes through A, B, and C must lie on both XL and YM—*i.e.*, must be at O.

Thus there is only one such circle. *Q.E.D.*

THEOREM 31

Equal chords of a circle are equidistant from the centre.

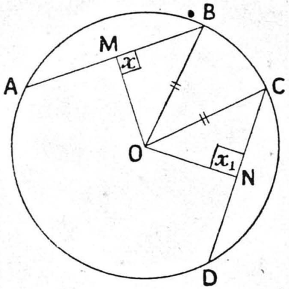


FIG. 275

Given AB, CD, two equal chords of the circle centre O, and OM, ON, perpendicular to AB, CD, respectively.

Prove that OM = ON.

Construction. Join BO, CO.

Proof. We know that

$$BM = \tfrac{1}{2}AB \quad (\text{perp}^{\text{r}} \text{ from centre to chord}),$$
$$CN = \tfrac{1}{2}CD \quad (\text{perp}^{\text{r}} \text{ from centre to chord}),$$
$$\text{and} \quad AB = CD \quad (\text{given}).$$
$$\therefore \quad BM = CN.$$

In the $\triangle$s OMB, ONC,

$$\begin{cases} BM = CN & (\text{just proved}), \\ OB = OC & (\text{radii}), \\ x = x_1 & (\text{rt. angles, given}). \end{cases}$$

$$\therefore \triangle OMB \equiv \triangle ONC$$

(two sides, non-included rt. angle),

and, in particular,

$$OM = ON.$$ *Q.E.D.*

THEOREM 32

Chords which are equidistant from the centre of a circle are equal.

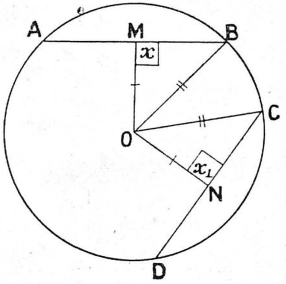


FIG. 276

Given AB and CD two chords of a circle centre O. OM, ON, are perpendicular to AB, CD, respectively, and OM = ON.

Prove that AB = CD.

Construction. Join OB, OC.

Proof. In the $\triangle$s OBM, OCN,

$$\begin{cases} OM = ON & \text{(given)}, \\ OB = OC & \text{(radii)}, \\ x = x_1 & \text{(rt. angles, given)}. \end{cases}$$

$\therefore$ $\triangle$ OBM $\equiv$ $\triangle$ OCN

(two sides, non-included rt. angle),

and, in particular,

BM = CN.

Now, BM = $\frac{1}{2}$AB (perp^r from centre to chord),

and CN = $\frac{1}{2}$CD (perp^r from centre to chord).

$\therefore$ AB = CD. *Q.E.D.*

Exercises: XXXVI

Section A

1. In a circle, radius 5 in., find the length of a chord which is (i) 3 in. from the centre, (ii) 4 in. from the centre, (iii) a in. from the centre.

2. In a circle, radius 13 in., find the distance from the centre to a chord of length (i) 10 in., (ii) 24 in., (iii) a in.

3. Find the diameter of a circle in which the distance from the centre to a chord of length 14 in. is (i) 24 in., (ii) $5\frac{1}{4}$ in., (iii) a in.

4. A rod PQ, length 6 ft., slides with its ends on a circular hoop (radius $3\frac{1}{4}$ ft.), centre S. M is the mid-point of PQ. Calculate SM. State the locus of M as the rod slides round the hoop.

5. Two equal circles have their centres 8 in. apart, and have a common chord 6 in. long. Find the radius of the circles.

6. Two parallel chords of a circle, radius 5 in., are 4 in. and 6 in. long. How far are they apart? (Two answers.)

7. An isosceles triangle, the base of which is 6 in. long, is described in a circle, radius 5 in. Calculate its area.

8. Prove that the area of a square inscribed in a circle of radius r in. is $2r^2$ sq. in.

9. Given that AB is a chord of a circle, centre S, with X the mid-point of AB and L any point on SX, prove that LX bisects angle ALB.

10. Prove that the line joining the mid-points of two parallel chords of a circle passes through the centre of the circle. Hence write down the locus of the mid-point of a variable chord of a fixed circle which moves parallel to a given line.

11. Two circles, centres A and B, cut at X and Y. A straight line is drawn through X parallel to AB to cut the circles at C and D. Prove that CD = 2AB.

12. Prove that when two circles intersect the line joining their centres bisects their common chord at right angles.

13. A straight line ABCD cuts two concentric circles at A, D, and B, C. Prove that AB = CD.

14. Prove that two intersecting chords of a circle cannot both be bisected by the point of intersection unless each is a diameter of the circle.

Section B

15. Show how to find the centre of a circle when an arc is given.

16. A chord, length $2a$ in., is placed in a circle, radius r in. Given that its distance from the centre is x in., show that $x = \sqrt{r^2 - a^2}$. Hence show that if two chords are drawn in a circle the greater chord is nearer the centre than the smaller.

17. If in Exercise 4, N is a point on the rod distance $2\frac{1}{16}$ ft. from P, write down the locus of N as the rod moves on the hoop.

18. Find the length of the equal sides of the triangle of Exercise 7.

19. Prove that the line joining the centres in Exercise 12 also bisects any chord drawn parallel to the common chord.

20. A circle through the ends of the base XY of a $\triangle$ XYZ cuts XZ, YZ, at L, M. The line joining Z to the centre of the circle bisects $\widehat{XZY}$. Prove that XL = YM.

21. Two equal chords AB and CD of a circle cut at X. Prove that AX = CX (or AX = DX).

22. A chord AB of a circle, centre O, is produced to C, making BC = AB. Prove that $OC^2 = OB^2 + 2BC^2$.

23. Given that in Fig. 277 ABC is a segment of a circle, radius r in., AL = LC = a in., BL = b in., and $\widehat{ALB} = 90°$, prove that

$$r = \frac{a^2 + b^2}{2b}.$$

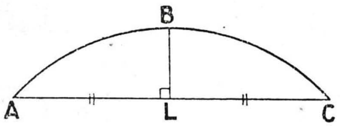


FIG. 277

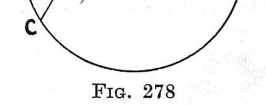


FIG. 278

24. In Fig. 278 S is the centre of the circle, AP = PS, and SPQ is an equilateral triangle. Prove that

$$PD = \left(\frac{\sqrt{13} + 1}{2}\right) AP.$$

ARC, SEGMENT, ANGLE IN A SEGMENT

The chord AB divides the circumference of the circle into two *arcs*, the arc AXB (the major arc AB) and the arc AYB (the minor arc AB).

The figure bounded by the arc AXB and the chord AB is called a *segment* of the circle. It is referred to as segment AXB (Fig. 279).

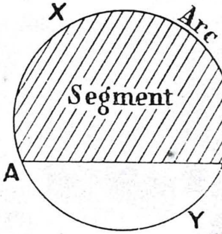


FIG. 279

In the circle in Fig. 280 P is any point on the arc AXB.

The angle APB is called an *angle in the segment* AXB. (There is no need to draw the chord AB.)

This angle APB is also called *an angle at the circumference standing on the arc* AYB, or *subtended by the arc* AYB.

The angles *a* and *b* are said to be *angles in the same segment*, or angles at the circumference *standing on the same arc* AYB, or angles at the circumference *subtended by the same arc* AYB.

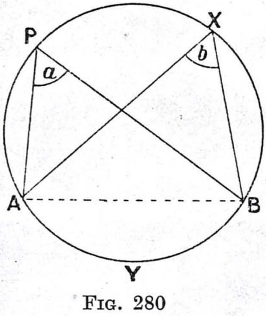


FIG. 280

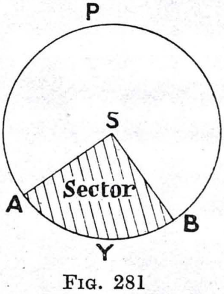


FIG. 281

The angle ASB in Fig. 281, in which S is the centre of the circle, is called the angle at the centre *standing on the arc* AYB, or *subtended by the arc* AYB.

The figure bounded by the two radii SA and SB and the arc AYB is called the *sector* ASBY.

ANGLE PROPERTIES OF A CIRCLE

Theorem 33

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

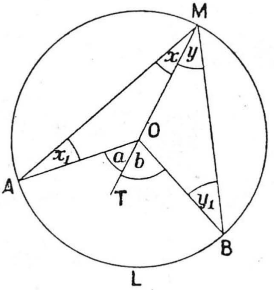


Fig. 282

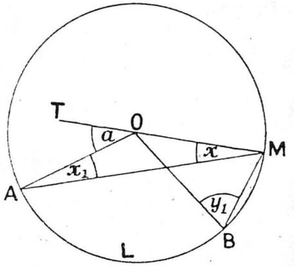


Fig. 283

In Fig. 283, as in Fig. 282, $\widehat{TOB}$ is denoted by b

and $\widehat{OMB}$ „ „ „ y.

Prove ALB an arc of a circle, centre O. M is any point on the remaining arc.

Given that $\widehat{AOB} = 2\widehat{AMB}$.

Construction. Join MO and produce to T.

In Fig. 282 T is inside $\widehat{AOB}$; in Fig. 283 T is outside $\widehat{AOB}$.

Proof. Since AO = MO (radii),

$\therefore \quad x = x_1$ (isosceles triangle).

Also $a = x + x_1$ (ext. angle theorem).

$\therefore \quad a = 2x$.

Similarly, $b = 2y$.

$\therefore \quad a + b = 2(x + y)$ in Fig. 282,

and $b - a = 2(y - x)$ in Fig. 283.

That is, $\widehat{AOB} = 2\widehat{AMB}$ in both figures. *Q.E.D.*

NOTES. (i) Notice that in Fig. 283 it is still convenient to denote $\widehat{TOB}$ and $\widehat{OMB}$ by single letters, but it would be *unwise* to show them in the figure, *since a confused figure would result.*

(ii) The theorem is still true, and the proof is the same as for Fig. 282, if the arc ALB is more than half the circumference. In this case the angle which ALB subtends at O is the reflex angle AOB (Fig. 284).

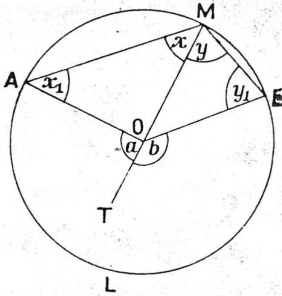


FIG. 284

THEOREM 34

Angles in the same segment of a circle are equal.

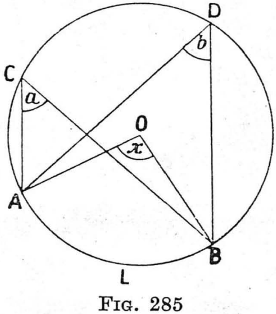


FIG. 285

Given a segment ACDB of a circle, centre O, with CA, CB, DA, and DB joined.

Prove that $a = b$.

Construction. Join AO, BO.

Proof. We know that $x = 2a$ (on same arc ALB).

Similarly, $x = 2b$ (on same arc ALB).

$\therefore \quad 2a = 2b.$

$\therefore \quad a = b.$ *Q.E.D.*

EXERCISES: XXXVII

Section A

1. Given that S is the centre of the circle ABC, calculate $\widehat{ASB}$ in degrees when

(i) $\widehat{BAC} = 60°$, $\widehat{ABC} = 80°$. (iii) $\widehat{BAC} = x°$, $\widehat{ABC} = y°$.

(ii) $\widehat{BAC} = 30°$, $\widehat{ABC} = 25°$.

2. If in Fig. 286 $c = 70°$ and $b = 30°$, find the value of d in degrees.

3. Find the value of e in Fig. 286 when $a = 75°$, $b = 40°$.

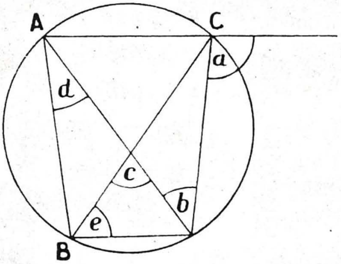


FIG. 286

4. Find which is greater, AB or AC in Fig. 286, when $b = 65°$, $e = 50°$, $c = 70°$.

5. Calculate the angles of a triangle ABC described in a circle, centre S, given that $\widehat{SAC} = 20°$, $\widehat{CAB} = 30°$, and CA lies inside $\widehat{SAB}$.

6. The circles CAE, DAF, meet at A and B, and CAD, EBF, are straight lines. Given that $\widehat{BCD} = 40°$ and $\widehat{CDB} = 30°$, calculate $\widehat{EAF}$.

7. The chords AB, CD, of a circle meet at K. Prove that triangles AKC, BKD, are equiangular. (Take K inside the circle.)

8. Given that S is the centre of a circle ABC and AS is parallel to CB, prove that $\widehat{ASC} = 2\widehat{SAB}$. (Take into account the 'sense' of AS and CB—*i.e.*, take the points of the quadrilateral ASBC in the order A, S, B, C.)

9. S is the centre of circle ABC. Find a relation between $\widehat{ASC}$ and $\widehat{SAB}$ when AS is parallel to BC.

10. Prove that the diagonals of a trapezium described in a circle are equal.

11. The bisector of $\widehat{BAC}$ of a triangle meets the circle ABC at D and meets BC at X. Prove that $\widehat{ABD} = \widehat{AXC}$.

12. The centre of a circle ABC is S, and S and C are on opposite sides of AB. Prove that $2\widehat{ACB} + \widehat{ASB} = 360°$.

13. In the $\triangle$ ABC, AB = AC, and the perpendicular from B to AC meets the circle ABC at D. Prove that $\widehat{BAC} = 2\widehat{CAD}$.

14. Two fixed circles intersect at A and B. A variable straight line through B cuts the circles at C and D. Prove that $\widehat{CAD}$ is constant.

Section B

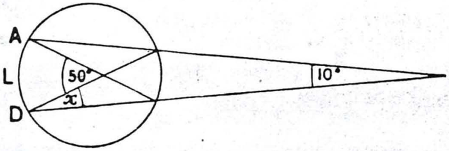


FIG. 287

15. Referring to Fig. 287, calculate (*a*) the value of x, (*b*) the angle which arc ALD subtends at the centre of the circle.

16. If in Fig. 288 AC = CD, CE = ED, $a = 30°$ and $b = 85°$, find which is greater, AB or BC.

17. Given that C and D are on the same side of AB and D is outside the circle ACB, prove that $\widehat{ADB} < \widehat{ACB}$.

18. The quadrilateral ABCD is described in a circle, its diagonals meet at E, and EF is drawn parallel to CD to meet AD at F. Prove that $\triangle$s ABC, EFD, are equiangular.

19. In the $\triangle$ ABC, AC > BC, a circle through A and B cuts AC at X and BC at Y, and BX cuts AY at L. Prove that AL > BL.

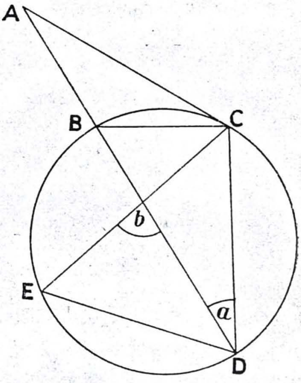


FIG. 288

20. In Fig. 289 LA || MB and BC = BA. Prove that $x = y$.

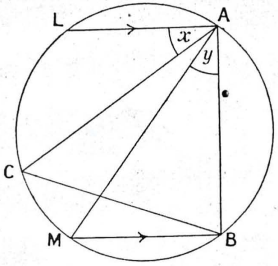


Fig. 289

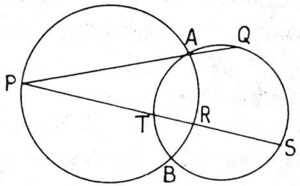


Fig. 290

21. In Fig. 290 B, R, and Q lie on a straight line. Prove that TQ = QS.

22. The altitude AD of a △ ABC is produced to meet the circle ABC at X, a point H is taken on DA such that DH = DX, and BH is produced to meet AC at E. Prove that the △s HBD, AHE, are equiangular. Deduce that **the three altitudes of a triangle meet in a point.**

CONCYCLIC POINTS AND CYCLIC QUADRILATERAL

If the points A, B, C, D, E, etc., are such that a circle can be drawn to pass through them, then A, B, C, D, E, etc., are said to be concyclic points (Fig. 291).

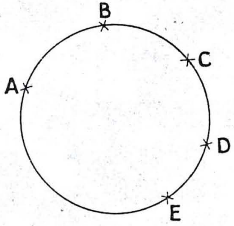


Fig. 291

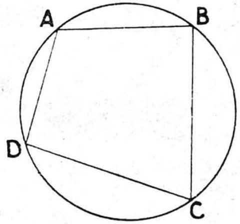


Fig. 292

If a quadrilateral ABCD is such that a circle can be drawn to pass through A, B, C, and D, then ABCD is a cyclic quadrilateral (Fig. 292).

THEOREM 35 (i)

The opposite angles of a quadrilateral inscribed in a circle are supplementary.

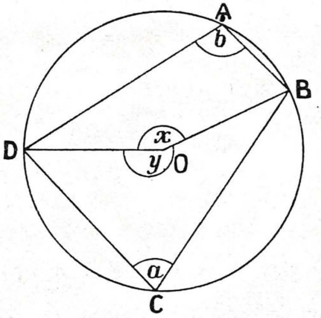


FIG. 293

Given a quadrilateral ABCD inscribed in a circle, centre O.

Prove that $a + b = 180°$.

Construction. Join BO, DO.

Proof. We know that

$$x = 2a \quad \text{(on same arc BAD)},$$

$$\text{and } y = 2b \quad \text{(on same arc BCD)}.$$

Adding, we get

$$x + y = 2(a + b).$$

$$\text{Now } x + y = 360° \quad \text{(one revolution)}.$$

$$\therefore 2(a + b) = 360°.$$

$$\therefore a + b = 180°.$$

Q.E.D.

THEOREM 35 (ii)

If one side of a quadrilateral inscribed in a circle is produced the exterior angle so formed is equal to the opposite interior angle of the quadrilateral.

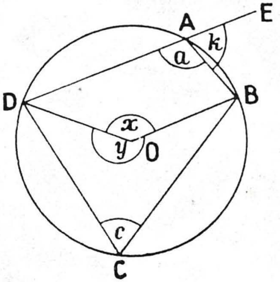


FIG. 294

Given a quadrilateral ABCD inscribed in a circle, centre O, and DA produced to E.

Prove that $k = c$.

Construction. Join BO, DO.

Proof. We know that

$$x = 2c \quad \text{(on same arc BAD)},$$

$$\text{and} \quad y = 2a \quad \text{(on same arc BCD)}.$$

Adding, we get

$$x + y = 2(c + a).$$

$$\text{Now} \quad x + y = 360^\circ \quad \text{(one revolution)}.$$

$$\therefore \quad 2(c + a) = 360^\circ.$$

$$\therefore \quad c + a = 180^\circ.$$

$$\text{Also} \quad k + a = 180^\circ \quad \text{(half a revolution)}.$$

$$\therefore \quad k + a = c + a.$$

$$\therefore \quad k = c.$$

Q.E.D.

• THEOREM 36

The angle in a semicircle is a right angle.

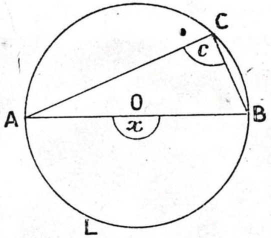


FIG. 295

Given ACB a semicircle, centre O.

Prove that $c = 90°$.

Construction. Complete the circle ACBL.

Proof. We know that

$$x = 180° \quad \text{(half a revolution)}$$

$$\text{and} \quad x = 2c \quad \text{(on arc ALB)}.$$

$$\therefore \quad 2c = 180°.$$

$$\therefore \quad c = 90°.$$

Q.E.D.

THEOREM 37

Any angle in a segment of a circle less than a semicircle is greater than one right angle, and any angle in a segment greater than a semicircle is less than one right angle.

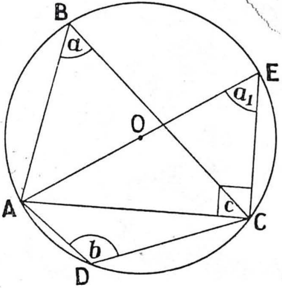


FIG. 296

Given $\widehat{ABC}$ in a segment of a circle greater than the semicircle ACE and $\widehat{ADC}$ in a segment less than a semicircle.

Prove that (i) $a < 90°$.
(ii) $b > 90°$.

Construction. Join AE, EC.

Proof. We know that

$a = a_1$ (on arc ADC)

and $c = 90°$ (angle in semicircle).

Since $a_1 + c < 180°$ (sum of $\angle$s of $\triangle = 180°$),

$\therefore\ a_1 + 90° < 180°$.

$\therefore\ a_1 < 90°$.

$\therefore\ a < 90°$. *Q.E.D.*

Also $a + b = 180°$ (cyclic quad[1] theorem),

and $a < 90°$ (proved).

$\therefore\ b > 90°$. *Q.E.D.*

EXERCISES: XXXVIII

Section A

1. In the cyclic quadrilateral ABCD, $\widehat{ACB} = 55°$ and $\widehat{CAB} = 60°$. Calculate $\widehat{CDA}$.

2. Given that in Fig. 297 $a = 97°$, $b = 45°$, and $c = 26°$, calculate x.

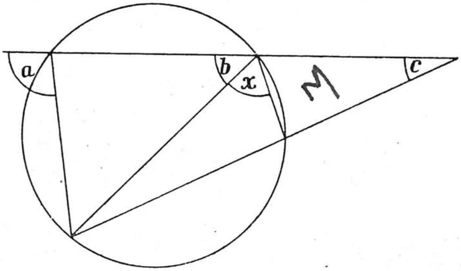


FIG. 297

3. Two circles cut at A and B. Straight lines CAD and LBM are drawn through A and B to meet the circles at C, D, and L, M. Given that $\widehat{CLM} = 70°$, calculate $\widehat{LMD}$ and prove that CL is parallel to DM.

4. Given that AB is a diameter of the circle ACBD and $\widehat{ABC} = 43°$, calculate $\widehat{CDB}$.

5. The circle on AB as diameter cuts the sides AC and BC of a $\triangle$ ABC at X and Y, and $\widehat{ACB} = 63°$. Calculate the angle between BX and AY.

6. A point L is taken on the side AC of a $\triangle$ ABC in which AB = BC such that $\widehat{ABL} = 22°$ and $\widehat{BLC} = 50°$, and BL meets the circle ABC at D. Which is greater, AD or BC?

7. A pentagon AXYZB is drawn in the circle on AB as diameter, and $\widehat{AXY} = 130°$. Calculate $\widehat{YZB}$.

8. Prove that a parallelogram described in a circle is a rectangle.

9. Given that the quadrilateral cut off from a triangle by drawing a line parallel to the base is cyclic, prove that the triangle is isosceles.

10. Two circles cut at A and B. Straight lines XAY and PBQ are drawn through A and B and are terminated by the circles at Y and Q. X and P are on the same circle. Prove that XP is parallel to YQ.

11. Prove that if in the previous exercise XY is parallel to PQ, then XY = PQ.

12. Given that the circles on AB and AC as diameters cut again at P, prove that BPC is a straight line.

13. The straight line AB is bisected at C. Circles are drawn on AB and AC as diameters. Prove that any chord through A of the first circle is bisected by the second circle.

14. The side AD of a cyclic quadrilateral ABCD is produced to E. The bisector of $\widehat{EDC}$ meets the circle ABC at L. Prove that BL bisects $\widehat{ABC}$.

15. Prove that the sum of three non-adjacent angles of a cyclic hexagon is four right angles.

Section B

16. The sides AB and DC of a cyclic quadrilateral ABCD meet at E, and the sides BC and AD are produced to meet at F. If $\widehat{E} = 40°$ and $\widehat{F} = 30°$, what is the value in degrees of $\widehat{BAD}$?

17. The diagonals of the quadrilateral ABCD meet at X. The circles ADX and BCX meet at a point on AB. $\widehat{DAB} = 70°$ and $\widehat{CBA} = 45°$. Calculate the value of $\widehat{AXB}$.

18. Prove that if ABCD is a cyclic quadrilateral in which AB and DC are produced to meet at E, and BC and AD are produced to meet at F so that $\widehat{DEF} = 2\widehat{AED}$ and $\widehat{BFE} = 2\widehat{AFB}$, then $\widehat{BAD} = 72°$.

19. In a circle, centre S, ABC is a minor arc. A second circle SADC is drawn through A, S, and C. Prove that

$$\widehat{ABC} = 90° + \tfrac{1}{2}\widehat{ADC}.$$

20. Two circles intersect at A and B. A straight line is drawn through A to cut the circles at C and D. DB (produced if necessary) cuts the first circle at L. Prove that $\widehat{ACB} = \widehat{LAC} - \widehat{ADB}$.

21. A circle is described on the side AB of a parallelogram

ABCD as diameter. The circle bisects BC at X. Prove that the area of the trapezium AXCD equals three-quarters of the area of the parallelogram ABCD.

22. A pentagon AXYZB is drawn in a circle on AB as diameter, and AX = XY and YZ = ZB. Prove that $\hat{Y} = \hat{A} + \hat{B}$.

23. The sides AB and DC of a cyclic quadrilateral ABCD meet at L, and sides BC and AD meet at M. The circles BCL and DCM meet at N. Prove that LNM is a straight line.

24. The bisector of the angle BAC meets the circle ABC at D, DX is drawn perpendicular to AB, and DY is perpendicular to AC. Prove that BX = CY.

Tests for Concyclic Points

Consideration of the examples given below and of No. 7 of Exercises: XXVI suggests the truth of the next two theorems.

Examples

Draw figures to satisfy the groups of data given below. In each case draw the circle through A, B, and C. Does the circle pass through D?

I. A quadrilateral ABCD, in which

(*a*) $\widehat{BAC} = 40°$, $\widehat{BDC} = 35°$.
(*b*) $\widehat{BAC} = 40°$, $\widehat{BDC} = 40°$.
(*c*) $\widehat{BAC} = 40°$, $\widehat{BDC} = 45°$.

II. A quadrilateral ABCD in which

(*a*) $\widehat{ABC} = 80°$, $\widehat{ADC} = 105°$.
(*b*) $\widehat{ABC} = 80°$, $\widehat{ADC} = 100°$.
(*c*) $\widehat{ABC} = 80°$, $\widehat{ADC} = 95°$.

THEOREM 38

If the line joining two points subtends equal angles at two other points on the same side of it the four points lie on a circle.

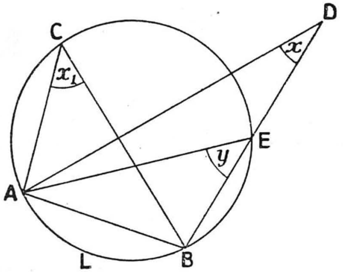


FIG. 298

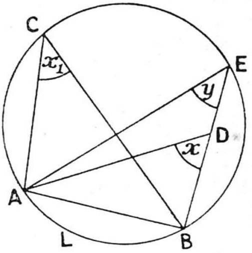


FIG. 299

Given two points A and B, and two points C and D on the same side of AB such that $x = x_1$.

Prove that A, B, C, and D lie on a circle.

Proof. Suppose the circle through A, B, and C does not pass through D.

Let it cut BD at E (Fig. 298), or let it cut BD produced at E (Fig. 299). Then $x = x_1$ (given),

and $x_1 = y$ (on arc ALB).

$\therefore\ x = y.$

This is impossible, for

$$\left.\begin{array}{l} y > x \text{ in Fig. 298} \\ x > y \text{ in Fig. 299} \end{array}\right\} \text{(ext. angle theorem).}$$

$\therefore$ the supposition is wrong.

$\therefore$ the circle through A, B, and C does pass through D.

Q.E.D.

The following note should be read *after* Theorem 45, and it should be observed that then there is no fault in 'sequence.'

NOTE. In the proof of Theorem 38 we have assumed that the circle through A, B, and C cuts BD or BD produced at a point E which is on the same side of B as D.

Two other possibilities should be considered.

(i) DB a tangent to the circle.

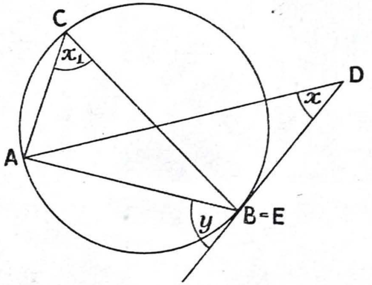


FIG. 300

In Fig. 300 $y = x_1$ (tangent, chord theorem),

and $x_1 = x$ (given).

$\therefore$ $y = x$,

which is impossible, as before (ext. angle theorem).

(ii) E lying on DB produced.

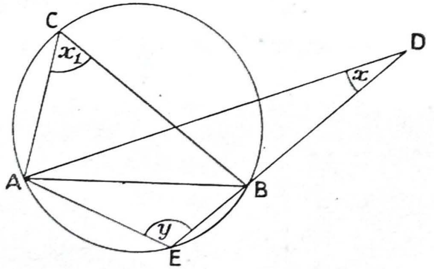


FIG. 301

In Fig. 301 $x = x_1$ (given),

and $y + x_1 = 2$ rt. angles (cyclic quad[l] ACBE).

$\therefore$ $y + x = 2$ rt. angles,

which is impossible, as y and x are only two angles of $\triangle$ AED.

Corollary. If a fixed line subtends an angle of constant size at a point which moves on one side of the fixed line, then the locus of the moving point is an arc of a circle passing through A and B.

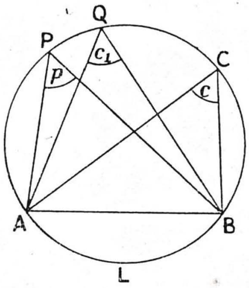


FIG. 302

For if C be a fixed point and Q a variable point such that angle ACB and angle AQB are each equal to the given angle, then Q lies on the arc ACB ($c = c_1$).

If P is any point on the arc ACB, then $p = c$ (on same arc ALB).

$\therefore$ P satisfies the locus conditions.

Hence the locus of the moving point is the arc ACB.

This locus property is frequently of use in constructions. Note that the actual arc ACB is drawn in Construction 16.

THEOREM 39

A quadrilateral in which two opposite angles are supplementary is cyclic.

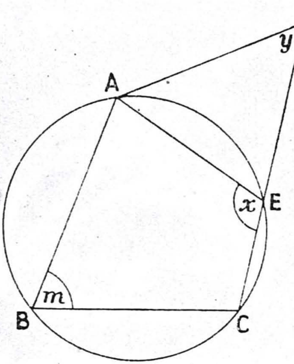


FIG. 303

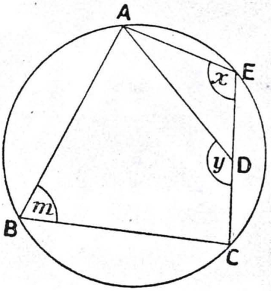


FIG. 304

Given a quadrilateral ABCD in which $m + y = 180°$.

Prove that A, B, C, and D lie on a circle.

Proof. Suppose the circle through A, B, and C does not pass through D.

Let it cut CD at E (Fig. 303), or let it cut CD produced at E (Fig. 304).* Then

$$m + y = 180° \quad \text{(given)},$$
$$\text{and} \quad m + x = 180° \quad \text{(cyclic quad}^{\text{l}}\text{ theorem)}.$$
$$\therefore \quad m + y = m + x.$$
$$\therefore \quad y = x.$$

This is impossible, for

$$\left.\begin{array}{l} x > y \text{ in Fig. 303} \\ y > x \text{ in Fig. 304} \end{array}\right\} \text{(ext. angle theorem)}.$$

$\therefore$ the supposition is wrong.

$\therefore$ the circle through A, B, and C does pass through D.

Q.E.D.

* To make the proof of this theorem rigorous the pupil should consider for himself, as in Theorem 38, two further cases which might arise: (i) DC a tangent to the circle; (ii) E on the other side of C from D.

THEOREM 40

If the line joining two points subtends a right angle at a third point, then this point will lie on the circle on the given line as diameter.

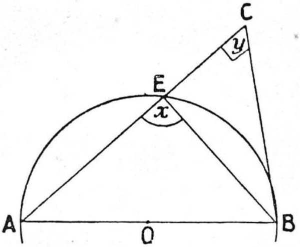


FIG. 305

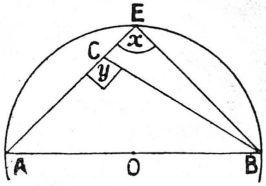


FIG. 306

Given two points A and B, and C a point such that $y = 90°$.

Prove that C lies on the circle on AB as diameter.

Proof. Suppose the circle on AB as diameter does not pass through C.

Let it cut AC at E (Fig. 305), or let it cut AC produced at E (Fig. 306).* Then

$$y = 90° \quad \text{(given)},$$

$$\text{and} \quad x = 90° \quad \text{(angle in semicircle)}.$$

$$\therefore \quad x = y.$$

This is impossible, for

$$\left.\begin{array}{l} x > y \text{ in Fig. 305} \\ y > x \text{ in Fig. 306} \end{array}\right\} \text{(ext. angle theorem)}.$$

$\therefore$ the supposition is wrong.

$\therefore$ the circle on AB as diameter does pass through C.

Q.E.D.

* Again the pupil should consider the other two possibilities—namely, (i) AC a tangent to the circle; (ii) E on the other side of A from C.

EXERCISES: XXXIX

Section A

1. Prove that the quadrilateral in Fig. 307 is cyclic.

2. Prove that the quadrilateral in Fig. 308 is cyclic.

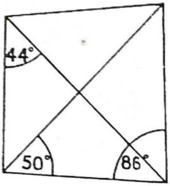


FIG. 307

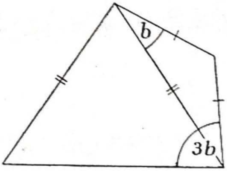


FIG. 308

3. In the quadrilateral ABCD, AC = AB, AD = DC, and $\widehat{CAD} = 35°$. Calculate the value of $\widehat{BDC}$.

4. In the △ ABC, $\widehat{ABC} = 50°$, and BE and CF are altitudes. Calculate the value of $\widehat{AEF}$.

5. Prove that if BY and CZ are altitudes of △ ABC, then BCYZ is cyclic.

6. Prove in Fig. 309 that if $\widehat{X} + \widehat{Y} + \widehat{Z} = 180°$, then the circles ABZ, ACY, and BCX have a common point.

7. Given that the quadrilateral ABCD has AB parallel to DC and $\widehat{A} = \widehat{B}$, prove that the quadrilateral is cyclic.

8. A circle drawn through the points B and C of a parallelogram ABCD cuts AB at X and DC at Y. Prove that ADYX is a cyclic quadrilateral.

9. A line drawn parallel to the side BC of a cyclic quadrilateral ABCD cuts AB at X and DC at Y. Prove that AXYD is cyclic.

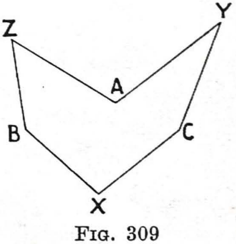


FIG. 309

10. Prove that the bisectors of the interior angles of any quadrilateral form a cyclic quadrilateral.

11. Given that CZ and BY are altitudes of the △ ABC and meet at H, prove that

(i) AZHY is cyclic.

(ii) $\widehat{ZAH} = \widehat{ZCB}$.

(iii) **AH is perpendicular to BC.**

12. In Fig. 310 AXY, APQ, and XPL, are straight lines. Prove that XYML is cyclic. Prove also that PLMQ is cyclic.

13. A straight line XYZ cuts the sides AB and AC of a △ ABC at Z and Y, and cuts BC produced at X. Prove that the circles CXY, AYZ, BXZ, and ABC meet in a point.

14. The line AB is the diameter of a circle, and AD is any chord. A perpendicular to AB cuts AB at E and AD at C. Prove that ECDB is cyclic.

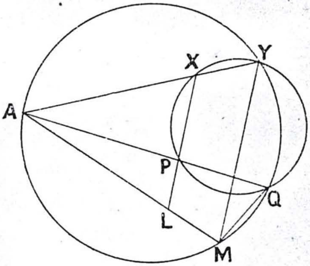


Fig. 310

15. Prove that the three feet of the altitudes of a triangle and the three mid-points of the sides of the triangle all lie on one circle.

Section B

16. Given in Fig. 311 that $a = 130°$ and $b = 50°$, prove that the centre of circle ABC lies on the arc ALC.

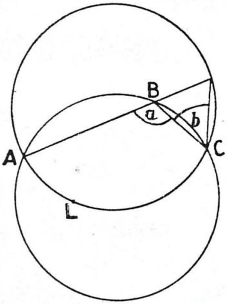


Fig. 311

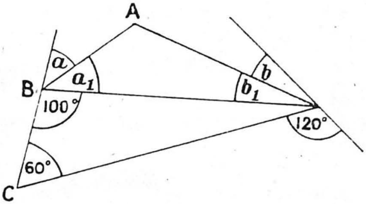


Fig. 312

17. Points X, Y, Z, are taken on the sides BC, CA, AB, of a △ ABC. The circles AZY, BZX, cut again at K. Prove that KYCX is cyclic.

18. Prove in Fig. 312 that if $a = a_1$ and $b = b_1$, then AB = BC.

19. In the △ ABC, BE and CF are altitudes and X is the midpoint of BC. Prove that $\widehat{FXE} = 2\widehat{FCE}$.

20. Prove that in the figure of the preceding exercise $\widehat{FEX} = \widehat{BAC}$.

21. Prove that the bisectors of the exterior angles of any quadrilateral form a cyclic quadrilateral.

ARCS

Theorem 41

Theorems connecting arcs, chords, and angles in equal circles are comprised under this heading.

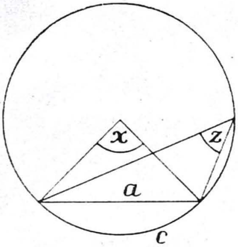


Fig. 313

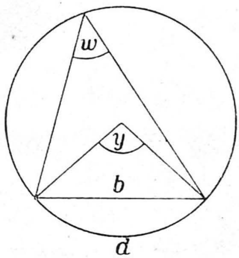


Fig. 314

If in equal circles

c and d are two minor arcs,
a ,, b ,, the chords which cut them off,
x ,, y ,, ,, angles they subtend at the centres,
z ,, w ,, ,, ,, ,, ,, ,, ,, circumferences,

then the following theorems are true:

Theorem 41 (i)

If $c = d$ then $a = b$, $x = y$, and $z = w$.

Theorem 41 (ii)

If $a = b$ then $c = d$, $x = y$, and $z = w$.

Theorem 41 (iii)

If $x = y$ then $a = b$, $c = d$, and $z = w$.

Theorem 41 (iv)

If $z = w$ then $a = b$, $c = d$, and $x = y$

Note. The same theorems are still true if the same circle be taken instead of equal circles.

Exercises: XL

Section A

1. Find the angle at the centre of a circle and the angle at the circumference subtended by one side of a regular inscribed

(*a*) 3-sided figure, (*d*) 8-sided figure,
(*b*) 4-sided figure, (*e*) 10-sided figure.
(*c*) 5-sided figure,

2. AB is the side of an equilateral triangle, and AC is the side of a square, each figure being inscribed in the same circle. Calculate the size of $\widehat{BAC}$ in degrees. (Two answers.)

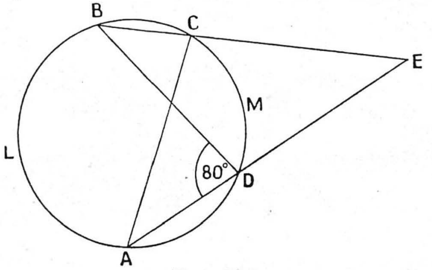


FIG. 315

3. In Fig. 315 AC = CE. Prove that arc ALB = 2 arc CMD.

4. Given that AB and CD are parallel chords of a circle, prove that AC = BD and AD = BC.

5. ABCD is a cyclic quadrilateral with AB = CD. Prove that AC = BD.

6. Two equal circles have a common chord AB. A straight line XBY is terminated by the circles. Prove that △ AXY is isosceles.

7. In the cyclic pentagon ABCDE, AB = BC, CD = DE, $\widehat{CAE}$ = 82°, and $\widehat{CEA}$ = 62°. Find the angle between BD and AC.

8. Prove that in Fig. 315 AB < 2CD.

9. In the circle AXBCY, X is the mid-point of the arc AXB, Y is the mid-point of the arc AYC, and XY cuts AB at L and AC at M. Prove that AL = AM.

10. Prove that if, in the cyclic hexagon ABCDEF, AB is parallel to ED and BC is parallel to FE, then CD is parallel to AF.

11. Two perpendicular intersecting chords of a circle divide the circumference into four arcs. Prove that the sum of one pair

of opposite arcs is equal to the sum of the other pair of opposite arcs.

12. Two equal non-intersecting circles CAD and BEF are cut by a third circle at A, D, and B, E. Given that CDEF is a straight line parallel to AB, prove that $\triangle$s ACD and BEF are congruent.

13. In the triangle ABC the side BC is fixed and $\widehat{BAC}$ is fixed in size. Prove that as A moves the bisector of $\widehat{BAC}$ always passes through a fixed point.

14. Two equal circles have a common chord AB. An equilateral $\triangle$ ABC is described with C on the first circle, and a straight line XAY is drawn outside this triangle cutting the first circle at X and the second circle at Y. Prove that XC = AY.

15. From the result of the previous exercise deduce that in the figure of that exercise XB = XA + XC.

Section B

16. A side of a regular pentagon inscribed in a circle is AB, and a side of a regular seven-sided polygon in the same circle is BC. Given that $\widehat{ACB}$ is obtuse, calculate the angles of the $\triangle$ ABC.

17. In the circle ABXCY . . . AB and BC are two sides of a regular pentagon, and AX and XY are two sides of a square. Find the ratio of the minor arc CY to the minor arc AB.

18. A hexagon ABCDEF is inscribed in a circle with AF parallel to BE and CF parallel to DE. Prove that AC = BD.

19. A point B is taken on the base of an isosceles $\triangle$ AXY. Prove that the circles AXB and AYB are equal.

20. In Fig. 316 ABC is an equilateral triangle, and a, b, c, etc., are arcs. Prove that $a + d = b + e = c + f$.

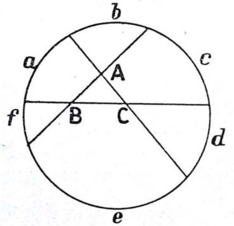


FIG. 316

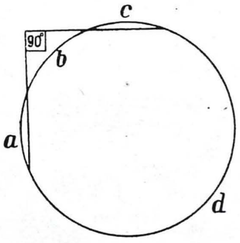


FIG. 317

21. In Fig. 317 a, b, c, and d are arcs of the circle. Prove that $a + 3b + c = d$.

22. In Fig. 318 the circles are fixed, and L moves on the arc ALB. ALX and BLY are straight lines. Prove that arc YX is of constant length.

23. In the △ ADC, AD = DC and $\widehat{DAC} = 3\,\widehat{ADC}$. B is a point on the major arc CD of the circle ADC such that the minor arcs BC, BD, are equal. Prove that △ ABC is isosceles.

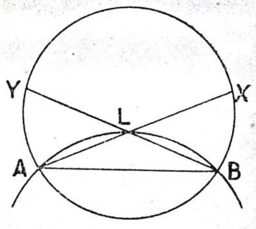


FIG. 318

TANGENT AND SECANT

A straight line which cuts a circle in two points is called a *secant* to the circle. In Fig. 319 XY is a secant.

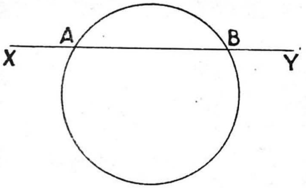


FIG. 319

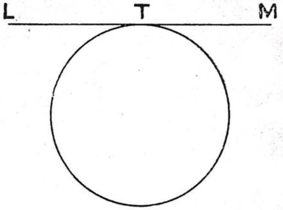


FIG. 320

A straight line which, however far it is produced, meets a circle in one point only is called a *tangent* to the circle. In Fig. 320 LTM is a tangent. The point T which is on both the tangent and the circle is called the *point of contact* of the tangent. A straight line which *touches* a circle is a tangent to the circle.

A tangent touches or meets a circle, but does not cut or intersect it.
A secant cuts or intersects or meets a circle, but does not touch it.

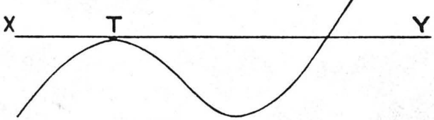


FIG. 321

XTY is a tangent to the curve in Fig. 321.

Does the above definition of a tangent hold for all curves?

TANGENT THEOREMS

THEOREM 42

The tangent to a circle at any point on its circumference is perpendicular to the radius of the circle which passes through the point of contact.

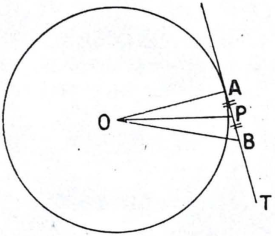


FIG. 322

Given OA a radius and AT a tangent to a circle.

Prove that $\widehat{OAT} = 90°$.

Proof. Suppose $\widehat{OAT}$ is not equal to 90°.

Draw OP perpendicular to AT.

Make PB = AP.

Join BO.

Thus we have made OP the perpendicular bisector of AB.

∴ OA = OB (first locus theorem).

∴ B also lies on the circle.

∴ the tangent AT cuts the circle in two points.

This is impossible (definition of a tangent).

∴ our supposition is wrong.

∴ $\widehat{OAT}$ *is* equal to 90°. *Q.E.D.*

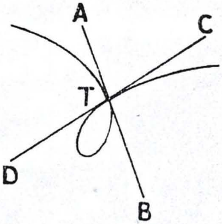


FIG. 323

Corollary. Only one tangent can be drawn to a circle to pass through a given point on its circumference.

In the curve in Fig. 323 ATB and CTD are both tangents at T to the curve.

THEOREM 43

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

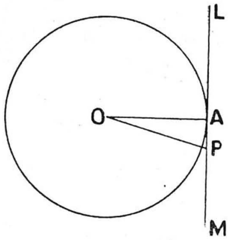


FIG. 324

Given OA a radius of a circle and LAM a line drawn perpendicular to it.

Prove that LAM is a tangent to the circle.

Construction. Take P any point on LM other than A, and join OP.

Proof. We know that

$$OA < OP \text{ (OA perpendicular to LM)},$$

and OA is a radius of the circle.

$\therefore$ P lies outside the circle

Since this is true of every point on LM other than A,

$\therefore$ LM meets the circle in one point only, namely A.

$\therefore$ LM is, by definition, a tangent to the circle.

Q.E.D.

LIMITS

It is assumed that the pupil will have had an explanation of the following statements:

(i) $\underset{x \to a}{\text{Limit}} \dfrac{x-a}{x-a} = 1.$

(ii) $\underset{x \to 2}{\text{Limit}} \dfrac{x^2-x-2}{x-2} = 3.$

(iii) $S = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots + \frac{1}{2^n}$ is always less than 2, however great n may be, and $(2 - S)$ approaches 0 as n increases without limit.

We now explain a 'limit' definition of a tangent to a circle at a given point on the circle, and this definition can be extended to curves other than the circle.

Consider a moving point P on a circle, centre O, with X a fixed point on the circle. Produce XP both ways, getting the line LXPA (Fig. 325).

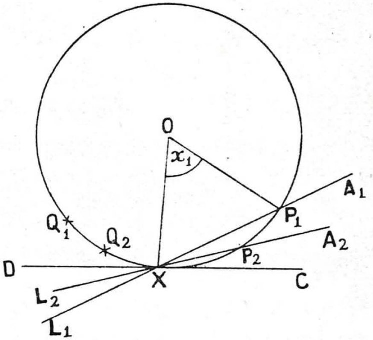


FIG. 325

The figure shows two positions (namely P_1, P_2) of the moving point P.

Let $\widehat{XOP} = x°$.

Then $\widehat{OXP} = \left(90 - \frac{x}{2}\right)°$ (since OX = OP).

Now let P start from the position P_1 and approach X, passing through P_2, etc.

As P approaches X, x approaches zero.

$\therefore$ $\widehat{OXP}$ approaches 90°.

$\therefore$ the line LXPA approaches the line DXC, where DXC is the perpendicular to OX through X.

We *define the tangent at* X *to the circle to be the limit of the line* LXPA *as the arc* PX *approaches zero.*

From the above we see that whether we approach X from P_1 through P_2 or from Q_1 through Q_2 we arrive at the same limiting position—namely, the line through X perpendicular to OX.

Thus, there is one, and only one, tangent at X *to the given circle, and this tangent is perpendicular to* OX.

Extension. Consider the curve shown in Fig. 326, consisting of the two branches XP_1A and XQ_1B.

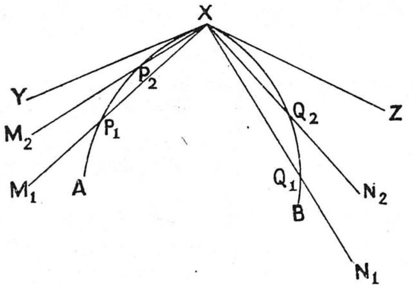


FIG. 326

Take any point P on the branch XP_1A. (Fig. 325 shows two positions of P—namely, P_1 and P_2.) Produce XP to M.

Let P start at P_1 and move along the branch AX to X. Then the limit of the line MPX as the arc PX approaches zero will in general be a definite line, say YX.

This line YX is defined to be a tangent at X to the curve.

Similarly, if the limit of the line XQN as the arc QX approaches zero is the line XZ, then XZ is also a tangent to the curve.

In this particular case there are two tangents, YX and XZ, at X to the curve.

EXAMPLE. Consider the cyclic quadrilateral ABCD in which the points B, C, and D are fixed and A moves (Fig. 327).

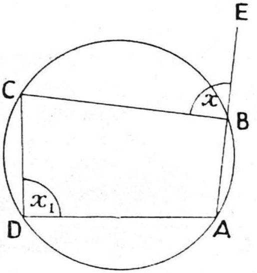


FIG. 327

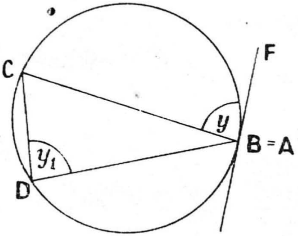


FIG. 328

Produce AB to E.

For all positions of A we know that $x = x_1$ (exterior angle of cyclic quadrilateral).

Now let A → B (Fig. 328).

∴ the line ABE → the tangent BF,

$$x \to y,$$

and $x_1 \to y_1$.

Since $x = x_1$,

$$\therefore \quad y = y_1.$$

Thus we have deduced Theorem 44 from Theorem 35.

EXERCISES: XLI

Section A

1. Prove that chords drawn parallel to a tangent to a circle are bisected by the diameter through the point of contact.

2. Prove that the line joining the points of contact of parallel tangents to a circle passes through the centre of the circle.

3. What is the locus of the centres of the circles which touch a given straight line at a given point?

4. *What is the locus of the centres of the circles which touch two given intersecting straight lines?*

5. Tangents at the points A and B on a circle, centre S, meet at T. Prove that $\triangle$s ATS and BTS are congruent.

6. Two circles have the same centre. Prove that all the chords of the outer circle which touch the inner circle are equal. Prove also that each chord is bisected at the point of contact.

7. Prove that all the equal chords of a given circle touch a fixed circle.

8. Perpendiculars AL and BM are drawn from the extremities o a diameter AB of a circle on to the tangent at P. Prove that LP = PM.

Section B

9. A point P is taken outside a circle, centre S. The radius SA of the circle is produced to L, making LA = AS. The circle on AP as diameter passes through L. Prove that the tangent at A to the original circle bisects SP.

10. Prove in Exercise 8 that $\widehat{LAP} = \widehat{PAB}$.

11. Chords AL and AM are drawn through the end of a diameter AB of a circle, and are produced to cut the tangent at B in P and Q. Prove that LMQP is cyclic.

12. In the triangle ABC, A is a fixed point, the area of the triangle is constant, and BC is of constant length. Prove that BC touches a fixed circle.

13. A triangle is drawn with its sides touching a circle, radius r in. Given that the perimeter of the triangle is $2s$ in., prove that its area is rs sq. in.

14. Find the locus of the points of contact of tangents from a fixed point to a system of concentric circles.

ALTERNATE SEGMENT

Fig. 329 shows a tangent CAD to a circle and a chord AB through the point of contact. If we are considering the angle a the segment AXB is called the *alternate* segment. If we are considering $\widehat{BAC}$ the segment AYB is the alternate segment.

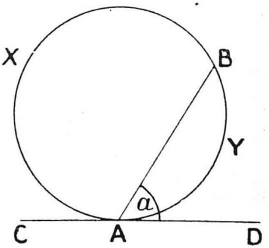


FIG. 329

*THEOREM 44

The angle between a tangent to a circle and a chord through the point of contact is equal to any angle in the alternate segment cut off by the chord.

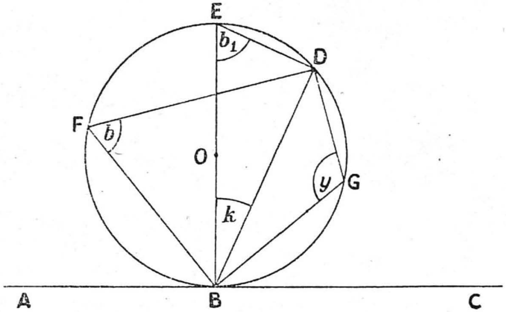


FIG. 330

Given a circle, centre O, a tangent ABC, and a chord BD, b is any angle in the segment alternate to $\widehat{DBC}$, and y is an angle in the segment alternate to $\widehat{DBA}$.

Denote $\widehat{DBC}$ by a and $\widehat{DBA}$ by x.

Prove that

(i) $a = b$.

(ii) $x = y$.

Construction. From B draw the diameter BOE. Join ED.

Proof. (i) Since $\widehat{BDE} = 90°$ (angle in semicircle),

$\therefore\ b_1 + k = 90°$ ($\widehat{BDE} + b_1 + k = 180°$).

Also $a + k = 90°$ (tgt. perp^{r} to radius).

$\therefore\ a + k = b_1 + k$.

$\therefore\ a = b_1$.

Now $b_1 = b$ (on same arc BGD).

$\therefore\ a = b$. *Q.E.D.*

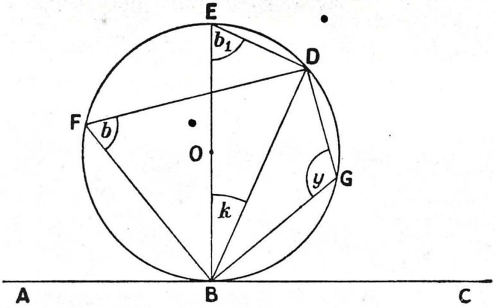


FIG. 330

In this figure $\widehat{DBC}$ is denoted by a and $\widehat{ABD}$ by x.

(ii) Since $b + y = 180°$ (cyclic quad[l]),

and $a + x = 180°$ (half a revolution),

$\therefore$ $a + x = b + y$.

Now $a = b$ (just proved).

$\therefore$ $x = y$. *Q.E.D.*

EXERCISES: XLII

Section A

1. If in Fig. 331 ABC is a tangent, what are the values in degrees of x and y?

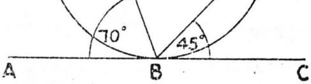


FIG. 331

2. In a circle, centre S, AB is a chord and AC a tangent. Given that $\widehat{BAC} = 84°$, find the value of $\widehat{ASB}$.

3. A tangent BC is drawn to a circle, BMA is an arc of this circle, and M and C are on the same side of BA. If $\widehat{AMB} = 118°$, what is the value of $\widehat{ABC}$?

4. The angle CAB of a △ ABC is 120°. A circle through C touches AB at A and cuts BC at D. Find the value of $\widehat{CDA}$.

5. The tangents at A and B to a circle meet at T, and $\widehat{ABT} = 46°$. Find the value of $\widehat{BAT}$.

6. In Fig. 332 TA and TB are tangents, and S is the centre of the circle. $\widehat{BAT} = 70°$. Find the value of angles *b*, *a*, and *c*.

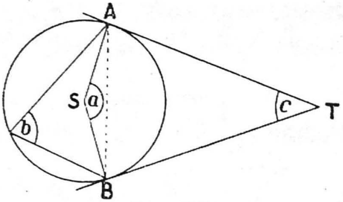


Fig. 332

7. The angles of a triangle whose sides touch a circle are 72°, 47°, and 61°. Find the angles of the triangle formed by joining the points of contact.

8. A quadrilateral ABCD is described in a circle, AD = DC, and the tangent at A makes an angle of 43° with AD. Calculate the value of $\widehat{ABC}$.

9. In Fig. 333 TA is a tangent. Given that AB = BC, prove that BA bisects $\widehat{TAC}$.

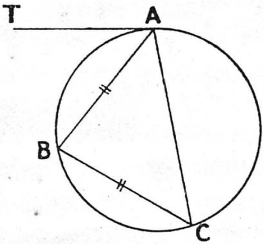


Fig. 333

10. Prove that the point of contact of a tangent to a circle bisects an arc cut off by a parallel chord.

11. In the $\triangle$ ABC the bisector of $\widehat{BAC}$ cuts BC at D. A circle through A touches BC at D and cuts AB, AC, at L, M. Prove that $\widehat{BDL} = \widehat{CDM}$.

12. A point P is taken on a circle of which AB is a diameter. AN is a perpendicular from A on to the tangent at P. Prove that PA bisects $\widehat{NAB}$.

13. Two circles cut at A and B. C is a point on one circle, and CA, CB, are produced to cut the second circle at L, M. Prove that LM is parallel to the tangent atC.

14. Circles are drawn through the vertex A of an isosceles triangle ABC to touch the base BC at B and C respectively. The circles cut again at X. Prove that $\widehat{BAX} = \widehat{CAX}$.

15. Two circles cut at A and B. A straight line touches one circle at C and the other circle at D. Prove that

$$\widehat{CAD} + \widehat{CBD} = 180°.$$

16. Two circles touch. Through their point of contact two straight lines are drawn and are terminated by the circles. Prove that the chords joining their extremities are parallel. Consider the cases of both internal and external contact.

17. Two circles touch internally at A, and a tangent at B to the inner circle cuts the outer circle at C and D. Prove that BA bisects $\widehat{CAD}$.

Section B

18. The tangents at A and B to a circle ABC meet at an angle of 40°, and the angle between the tangents at B and C is 72°. Calculate the size of $\widehat{ABC}$.

19. Two circles PAB and QAB meet at A and B. PAQ is a straight line. The tangents at P and Q meet at T, $\widehat{TPQ} = 72°$, and $\widehat{TQP} = 47°$. Calculate $\widehat{PBQ}$.

20. The diagonals of a cyclic quadrilateral ABCD meet at P, and AD and BC are produced to meet at Q. Given that $\widehat{APB} = 80°$ and $\widehat{AQB} = 10°$, find by calculation the angle between the tangents at A and B to the circle ABC.

21. The diagonals of a cyclic quadrilateral ABCD meet at X, and XT is a tangent to the circle ABX. Prove that XT is parallel to DC.

e tangent at A to a circle cuts the chord BC, produced, ›ve that $\triangle$s TAB and TAC are equiangular.

tangent at P to a circle cuts the diameter AB, produced, ›D is drawn perpendicular to AT. Prove that PA and PB ›ectors of the interior and exterior angle DPT.

24. Two segments of circles APB, AQB, lie on the same side of the common chord AB. Given that BQP is a straight line, prove that the angle between the tangents at A to the circles is equal to $\widehat{PAQ}$.

25. A point Z is taken on the tangent XZ to a circle, XY is a chord of the circle, and ZPQ is a straight line parallel to XY cutting the circle at P and Q. Prove that $\triangle$s XZP and XYQ are equiangular.

26. A point P is taken on the base BC of $\triangle$ ABC. The circles ABP, ACP, are drawn, and the tangents at B, C, meet at T. Prove that BACT is cyclic.

27. Prove that in the figure of the previous exercise one of the angles between the tangents at A to the two circles is equal to $\widehat{BAC}$.

28. The $\triangle$ ABC has $\widehat{ABC} = 90°$. The circle on AB as diameter cuts AC at D. The tangent at D to the circle ABD cuts BC at E. Prove that BE = EC.

29. A pentagon ABCDE is described in a circle. Given that AB is parallel to ED and BC is parallel to AE, prove that CD is parallel to the tangent at A.

30. In Fig. 334 ABCD is a semicircle. The tangents at B and C meet at T. Prove that TB = TE.

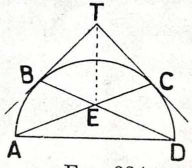


FIG. 334

Theorem 45

If a straight line be drawn from the extrem chord of a circle making with the chord an angle an angle in the alternate segment, then the tangent to the circle.

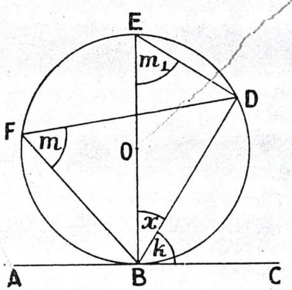


Fig. 335

Given a circle BDF, centre O, with ABC drawn making $\widehat{DBC} = \widehat{BFD}$.

Prove that ABC is a tangent to the circle.

Construction. Draw the diameter BOE and join ED.

Proof. Since $\widehat{EDB} = 90°$ (angle in semicircle),

$$\therefore \quad m_1 + x = 90° \quad (m_1 + x + \widehat{EDB} = 180°).$$

$$m_1 = m \quad \text{(on same arc BD).}$$

$$\text{Also} \quad m = k \quad \text{(given).}$$

$$\therefore \quad m_1 = k.$$

$$\therefore \quad k + x = 90° \quad (m_1 + x = 90°).$$

$\therefore$ ABC is a tangent to circle BDF (tangent, radius theorem).

Q.E.D.

Note. In the above proof k has been taken as acute. How should the proof be modified when k is obtuse ?

EXERCISES: XLIII

Section A

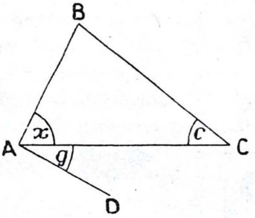


FIG. 336

1. State whether AD in Fig. 336 is a tangent to the circle through A, B, and C when

(i) $g = 60°$, $x = 40°$, $c = 70°$.
(ii) $g = 70°$, $x = 20°$, $c = 90°$.
(iii) $g = 25°$, $x = 92°$, $c = 63°$.
(iv) $g = 49°$, $x = 37°$, $c = 95°$.

2. Prove that if the parallelogram ABCD has AC = CD, then CB touches circle ACD.

3. Points E, F, are taken on the sides AB, AC, of a △ ABC such that EF ∥ BC. Prove that the circles AEF, ABC, touch at A.

4. In the quadrilateral ABCD, AC bisects $\widehat{BCD}$ and AD touches circle ABC. Prove that AB touches circle ACD.

5. In the circle, centre S, AS and BS are perpendicular radii, N is any point on AS, and BN is produced to cut the circle at C. Prove that BA touches circle ACN.

6. The diagonals of a parallelogram ABCD meet at E. Prove that circles AED and BEC touch.

7. The altitudes AD and BE of the △ ABC meet at H. Prove that if AB = AC, then BC touches the circle ABH.

8. In the quad[1] ABCD, BC is parallel to AD and BD is produced to meet circle ABC at E. Prove that AC touches circle ADE.

9. Two equal circles ABC and ABDE cut at A and B. CAE is a straight line, $\widehat{CBE} = 140°$, and $\widehat{BAD} = 20°$. Prove that ED touches circle EBC.

10. Prove that if BAC is an arc of a circle, E the mid-point of the remaining arc, and AE meets BC at D, then EC touches circle ACD.

Section B

11. A point D is taken on the base AC of a △ ABC. The tangent at D to circle BCD meets circle ABD at E. Prove that EA touches circle ABC.

12. In the △ ABC, AB = BC and D is a point on AC such that AD = DB. Prove that AB touches circle BDC.

13. In the regular pentagon ABCDE the lines BD and CE meet at F. Prove that BC touches circle BFE.

14. In the regular seven-sided figure ABCDEFG, AC and BE meet at H. Prove that AB touches circle AHE.

15. The sides BA and CD of a quad^l ABCD are parallel, and the circle BAD cuts BC at L. Prove that AD touches circle LCD.

16. The △ ABC has AC = AB, and D is any point on AB. The circle EBD touches CB at B. The circle ADF touches circle EBD at D. Prove that circle ADF touches circle BAC at A.

17. Two circles with centres A and B touch at C. A common tangent touches the circles at D and E. Prove that AD touches circle DEC.

18. In the △ ABC, AB = AC. L is any point on the minor arc AC of circle ABC, and AL and BC are produced to meet at M. Prove that AC touches circle LCM.

19. Two equal circles cut at L and M. A straight line ABCD meets one circle at A and B and the second at C and D. Given that AB = CD, prove that circles ALD and BLC touch at L.

20. The circle on AL as diameter passes through the points B, C, and D. DF is drawn parallel to AB to cut BC at F. O is the centre of the circle FDC. Prove that O lies on DL.

External and Internal Contact of Circles

If two circles meet at a point P, and if a line PA is a tangent to both circles, the two circles are said to *touch* at P.

If the centres of the touching circles are on opposite sides of AP the circles are said to touch *externally*, or to have *external contact* (Fig. 337).

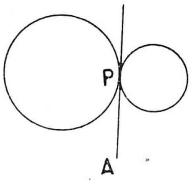


Fig. 337. External Contact

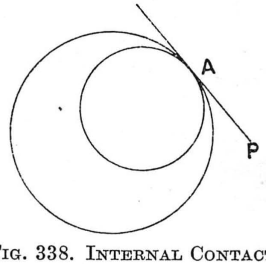


Fig. 338. Internal Contact

If the centres of the touching circles are on the same side of AP the circles are said to touch *internally*, or to have *internal contact* (Fig. 338).

THEOREM 46

If two circles touch the point of contact lies on the straight line through the centres.

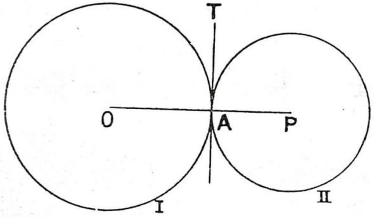


FIG. 339

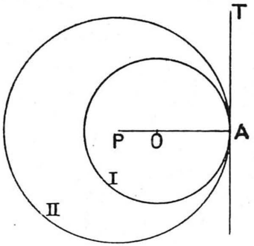


FIG. 340

Given two circles, centres O and P, touching at A.

Prove that A lies on the straight line through O and P.

Construction. Draw the common tangent AT.

Proof. We know that

$\widehat{OAT} = 90°$ (tgt., rad. theorem; circle (i))

and $\widehat{PAT} = 90°$ (tgt., rad. theorem; circle (ii)).

Therefore in Fig. 339 and Fig. 340 O and P both lie on the straight line through A perp^r to AT.

Therefore O, A, and P lie on a straight line;

i.e., A lies on the straight line through O and P. *Q.E.D.*

NOTE

(i) When the circles touch externally we have

$$OP = OA + AP,$$

i.e., OP = *sum of the radii.*

(ii) When the circles touch internally we have

$$OP = PA \sim OA,$$

i.e., OP = *difference of the radii.*

THEOREM 47

If two tangents are drawn to a circle from an external point, then

(i) they are equal in length;
(ii) the angle between them is bisected by the line joining the external point to the centre of the circle;
(iii) the same line bisects the angle between the radii drawn to the points of contact.

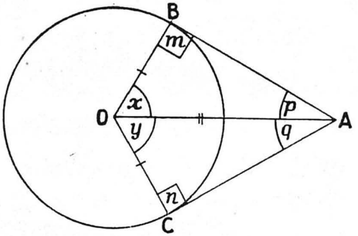


FIG. 341

Given a circle, centre O, and AB and AC tangents from A.

Prove that (i) AB = AC,

(ii) $p = q$,

(iii) $x = y$.

Proof. In the $\triangle$s OAB, OAC

$$\begin{cases} OA = OA \\ OB = OC & \text{(radii),} \\ m = n & \text{(rt. angles, tgt. perp}^{\text{r}}\text{ rad.).} \end{cases}$$

$\therefore$ $\triangle$ OAB $\equiv$ $\triangle$ OAC (two sides, non-included rt. angle), and, in particular,

(i) AB = AC,

(ii) $p = q$,

(iii) $x = y$. *Q.E.D.*

Exercises: XLIV

Section A

1. Find the length of the tangents drawn to a circle of radius 3 in. from a point 5 in. from the centre.

2. Find the length of the chord of contact in the previous exercise.

3. A point is 7 in. from the centre of a circle. The tangents from the point to the circle are 5 in. long. Find the radius of the circle and the length of the chord of contact.

4. Two tangents are drawn to a circle, from an external point, and one tangent makes an angle of 123° with the chord of contact. Find the angle between the tangents.

5. Tangents are drawn from an external point to a circle of radius 6 in. making the chord of contact 6 in. long. Find the lengths of the tangents.

6. Circles of radii 2 in., $2\frac{1}{2}$ in., and 3 in. touch externally. Find the distances between their centres.

7. Two circles, centres A and B and of radii 2 cm. and 3 cm., touch externally. A third circle, centre C and radius 7 cm., touches each circle internally. Find the lengths of the sides of the △ ABC.

8. Two circles of radii 7 cm. and 2 cm. have their centres 13 cm. apart. Find the length of their *direct* common tangent.

9. The length of the *transverse* common tangent of two circles of radii 4 cm. and 6 cm. is 10 cm. Find the distance between their centres.

10. The centres of three circles which touch externally are at the vertices of a triangle whose sides are 7 cm., 8 cm., and 11 cm. in length. Find the radii of the circles.

11. Each of four equal circles of radii 2 in. touches two of the other circles. Find the radius of the circle which touches each (*a*) externally, (*b*) internally.

12. Three equal circular disks are placed without overlapping in an equilateral triangle whose sides are 9 cm. long. Find their radius, given that each disk touches the other two disks.

13. A point P is taken on the common tangent drawn at A where two circles touch. Tangents PB and PC are drawn touching the circles at B and C respectively. Prove that PB = PC.

14. Prove that if the sides of a parallelogram touch a circle the parallelogram is a rhombus.

15. The quadrilateral ABCD is circumscribed to a circle. Prove that AB + CD = BC + DA.

16. Given that in the previous exercise S is the centre of the circle, prove that $\widehat{ASB} + \widehat{CSD} = 180°$.

17. AB and CD are the direct common tangents of two unequal circles, touching one circle at A and C and the other at B and D. Prove that AB = CD.

18. Prove that in the figure of the preceding exercise AC is parallel to BD.

19. A tangent to a circle, centre S, meets two parallel tangents at P and Q. Prove that $\widehat{PSQ} = 90°$.

20. Two circles touch at A, and LR and MS are parallel diameters of the circles. Prove that LAS is a straight line.

21. Prove that if the distance between the centres of two circles is equal to the sum of their radii, then the circles touch.

22. Two equal circles touch at A. A straight line through A meets one circle at B and the other at C. Prove that BA = AC.

23. Three equal circles touch externally at X, Y, and Z. XY and XZ are produced to meet the third circle again at P and Q. Prove that PQ is a diameter of the third circle.

24. A fixed point A is taken outside a fixed circle, centre S. Tangents from A touch the circle at P and Q. A variable tangent cuts AP and AQ (not produced) at X and Y. Prove that $\widehat{XSY}$ is fixed in size.

25. Two circles, centres P and Q, touch externally at A. A common tangent touches the circles at B and C respectively. Prove that the circle on PQ as diameter touches BC.

Section B

26. Two circles touch at B, AB is a diameter of one circle, and a tangent from A to the second circle touches the second circle at T. Prove that $2\widehat{ATB} + \widehat{TAB} = 90°$.

27. Two circles touch at A. A straight line touches one circle at B and touches the other at C. Prove that $\widehat{BAC} = 90°$.

28. What is the locus of a point from which a tangent to a fixed circle has a fixed length?

29. What is the locus of the centre of a variable circle which touches two equal non-intersecting fixed circles?

30. The sides of a △ ABC are AB = 4 cm., BC = 5 cm., and CA = 7 cm. Circles, centres C and A, touch externally and each

touches internally a circle, centre B. Find the radii of the three circles.

31. Two equal circles touch externally at A. A straight line touches one circle at P and the second at Q, and PQ = 8 cm. Find the radius of the circle which touches the two given circles and also touches PQ.

32. Two equal circles are placed without overlapping in a quare with 3-in. sides. Find their greatest radius.

33. A point P is taken 2 in. from the centre of a circle, centre S, radius 1 in., and a tangent PA is drawn touching the circle at A. A tangent PB is drawn to touch a concentric circle at B. Calculate the value of $\widehat{SBA}$.

34. Two tangents from an external point to a circle are perpendicular. If the chord of contact is 18 cm. long, what is the radius of the circle?

35. A circle, centre S, has a diameter ASB. Given that a circle is drawn touching the given circle and also touching the circles drawn on AS and BS as diameters, prove that its radius is equal to $\frac{1}{6}$AB.

36. A quadrilateral is circumscribed to a circle, and one of its sides subtends a right angle at the centre of the circle. Prove that the quadrilateral is a trapezium.

37. Tangents AB and AC are drawn from A to a circle, centre S, and a line is drawn through A perpendicular to CA to meet SB produced at K. Prove that KS = KA.

38. D is the mid-point of the hypotenuse AC of a right-angled △ ABC. Prove that the circle, centre D and radius equal to $\frac{1}{2}$(AB + BC), touches the circles on AB and on BC as diameters.

39. Two equal circles touch externally at A, AB is a diameter of one circle, and a tangent from B to the second circle cuts the first circle at C and touches the second circle at D. Prove that BC = 2CD.

40. A system of circles touch at A. A straight line through A cuts the circles at B, C, D, . . . etc. Show that the tangents at B, C, D, . . . etc., are all parallel.

41. A circle inscribed in a △ ABC touches AB at D. Prove that 2AD = AB + CA − BC.

42. Prove that if a polygon with an even number of sides circumscribes a circle the sums of its alternate sides are equal.

43. Two circles touch externally at D. A point B is taken on one circle and a point C is taken on the other circle. The tangents at B and C meet at A. Prove that $2\widehat{BDC} + \widehat{BAC} = 360°$.

Exercises: XLV

Locus Exercises

1. Circles are drawn with a fixed point as centre. Tangents are drawn from a second fixed point. Find the locus of their points of contact.

2. Through a fixed point outside a given circle lines are drawn to cut the circle. Prove that the locus of the mid-points of the chords cut off is *part* of a circle.

3. In the $\triangle$ ABC, A and B are fixed and $\widehat{C}$ is fixed in size. The bisectors of $\widehat{ABC}$ and $\widehat{BAC}$ meet at X. What is the locus of X?

4. In the $\triangle$ ABC, A and B are fixed, $\widehat{C}$ is fixed in size, and AC is produced to D, making DC = CA. Find the locus of D.

5. With the data of the previous exercise, AB is produced to E. The bisectors of $\widehat{CAB}$ and $\widehat{CBE}$ meet at Y. What is the locus of Y?

6. In the $\triangle$ ABC, A and B are fixed and $\widehat{C}$ is fixed in size. The altitudes of the triangle meet at H. What is the locus of H?

7. A straight thin rod slides with its ends on two fixed perpendicular lines. What is the locus of the mid-point of the rod?

8. Through a fixed point A a variable line AB is drawn, and through a fixed point C a parallel line CD is drawn. The bisectors of $\widehat{BAC}$ and $\widehat{DCA}$ meet at X. Find the locus of X.

9. The $\triangle$ ABC is fixed. Variable points L and M are taken on AB and AC (not produced), and BM and CL meet at X on the circle ALM. What is the locus of X?

10. The point P moves on a fixed circle which passes through two fixed points A and B. Straight lines PA and PB are drawn cutting a second fixed circle through A and B in L and M. If LB meets MA in X, what is the locus of X?

11. Fixed points A and B are taken on a fixed circle. A straight rod LM moves with its ends on the circle and on the same side of AB. AL and BM (produced if necessary) meet at X. Prove that the locus of X consists of two circular arcs.

12. If in Exercise 11 the restriction that L and M must be on the same side of AB be removed, what is now the locus of X?

13. The straight lines OA and OB are fixed at right angles. Points P and Q move on OA and OB so that PQ is of fixed length. The perpendiculars at P and Q to OA and OB meet at X. Find the locus of X.

14. Repeat Exercise 13 when OA and OB are not at right angles.

CONSTRUCTIONS

CONSTRUCTION 10

To construct a tangent to a circle through a given point on the circle.

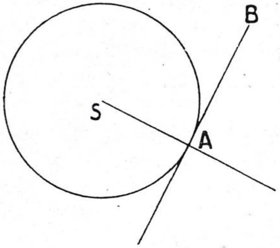


FIG. 342

Given a point A on a circle, centre S.

Construct the tangent at A to the circle.

Construction. Make $\widehat{SAB} = 90°$.
Then AB is the required tangent.

Proof. Since $\widehat{SAB} = 90°$,

$\therefore$ AB is the tangent to the circle at A
(tangent, radius theorem).

CONSTRUCTION 11

To construct the circum-circle of a given triangle.

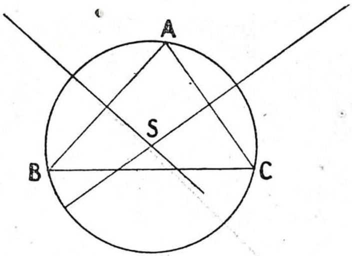


FIG. 343

Given a triangle ABC.

Construct the circum-circle of the △ ABC.

Construction. Construct the right bisectors of AB and AC to meet at S.

With centre S and radius SA, describe a circle.

This is the required circle.

Proof. We know SA = SB (S is on the rt. bisector of AB).

∴ the circle passes through B.

Also SA = SC (S is on the rt. bisector of AC).

∴ the circle passes through C.

∴ the circle passes through A, B, and C.

CONSTRUCTION 12: INSCRIBED CIRCLE

To construct a circle to touch the three sides of a given triangle internally.

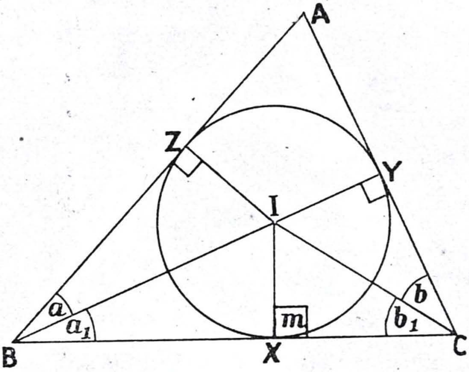


FIG. 344

Given a $\triangle$ ABC.

Construct a circle to touch AB, BC, and CA internally.

Construction. Draw the bisectors of the angles ABC and ACB to meet at I. Draw IX, IY, IZ, perpendicular to BC, AC, AB, respectively.

With centre I and radius IX draw a circle.

This is the required circle.

Proof. Since $b = b_1$,

$\therefore$ IX = IY (Second Locus Theorem).

Since $a = a_1$,

$\therefore$ IX = IZ (Second Locus Theorem).

$\therefore$ the circle with centre I which passes through X passes through Y and Z.

Also $m = 90°$ (construction).

$\therefore$ BC touches the circle (tangent, radius theorem).

Similarly, CA and AB touch the circle.

$\therefore$ the circle XYZ is the required circle.

Escribed Circle. In the same way, if we bisect the angles CBZ and YCB (see Fig. 345) and let the bisectors meet at I_1,

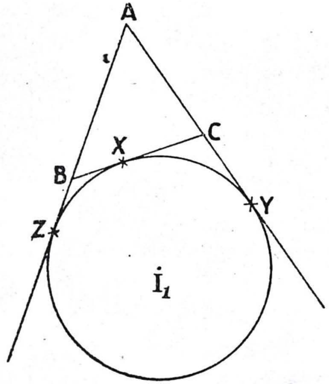


FIG. 345

we may prove that a circle can be drawn with centre I_1 to touch BC, AB produced, and AC produced.

Such a circle is called an *escribed circle* of the triangle ABC, and I_1 is called an *excentre* of the triangle. Two other escribed circles of the triangle may be drawn—namely,

(i) a circle touching AC, BA produced, and BC produced;
(ii) a circle touching AB, CA produced, and CB produced.

EXERCISES: XLVI

1. Show how to construct a circle to pass through two given points and have its centre on a given line.

2. Draw a triangle with sides 2 in., $2\frac{1}{2}$ in., and $3\frac{1}{2}$ in. in length, and draw its circum-circle.

3. Find the centre of a circle of which an arc is given.

4. Draw a triangle with sides 4 in., $3\frac{1}{2}$ in., and 3 in. in length. Draw the inscribed circle.

5. Take two points A and B 7 cm. apart. Draw a circle with ˑ 6 cm. to pass through A and B.

ˑ a triangle with sides 3 in., $2\frac{1}{2}$ in., and 2 in. in length. ᵔribed circle which touches the shortest side

7. Draw the quadrilateral ABCD with AB = 2 in., $\hat{A} = 100°$, $\hat{B} = 70°$, BC = 3 in., and $\hat{C} = 80$. Draw the circum-circle of △ ABC. Verify that the circle passes through D. (*Cf.* Theorem 39.)

8. Draw two lines ASB and CSD cutting at an angle of 70°, with AS = 4 cm., SB = 5 cm., CS = 2 cm., and SD = 6 cm. Draw the circle through ABC. Does it pass through D?

9. Repeat the previous exercise making AS = 2 in., SB = 1·8 in., CS = 2·4 in., and SD = 1·5 in. Does the circle through ABC now pass through D? (*Cf.* Theorem 51.)

10. Draw a quadrilateral at random. Draw a circle touching three of its sides. Verify that the circle does not touch the fourth side.

11. Draw a quadrilateral ABCD with AB = 5 cm., $\hat{B} = 85°$, BC = 4·5 cm., CD = 5·5 cm., and AD = 6 cm. Draw a circle to touch AD, AB, and BC. Does it touch CD? (*Cf.* Exercises XLIV, Number 15.)

12. Show how to construct a circle to pass through a given point and touch a given straight line at a given point.

13. Show how to construct a circle to touch two given parallel straight lines and to touch a third straight line.

14. Show how to construct a circle to touch two given parallel straight lines and to pass through a given point.

15. Draw a straight line AB. Take a point C 2 in. from AB. Draw a circle of radius $1\frac{1}{2}$ in. to pass through C and touch AB.

16. Make $\widehat{BAC} = 50°$. Draw a circle, radius 4 cm., to touch AB and AC.

17. Find the centre of a circle which cuts off from the sides of a given triangle three equal chords.

18. Draw a regular pentagon, with 1-in. sides. Draw a circle touching the five sides.

19. Inscribe a circle in a given rhombus.

20. The radius of the in-circle of a right-angled triangle is r in. and the hypotenuse is a in. long. Prove that

(i) the perimeter is $2(a + r)$ in. long;
(ii) the area of the triangle is $r(a + r)$ sq. in.

Hence, or otherwise, show how to construct a right-angled triangle given the lengths of the hypotenuse and the radius of the in-circle.

CONSTRUCTION 13

To construct two tangents to a circle from a given external point.

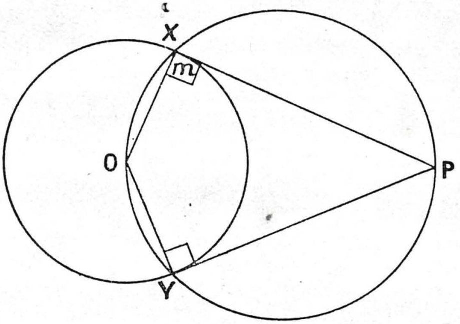


FIG. 346

Given a circle, centre O, and P a point outside the circle.

Construct two tangents from P to the circle.

Construction. On OP as diameter, draw a circle to cut the circle centre O in X and Y. Join PX and PY.

Then PX and PY are the required tangents.

Join OX and OY.

Proof. We know that $m = 90°$ (angle in a semicircle).

$\therefore$ PX is a tangent to the circle (tangent, radius theorem).

Similarly PY is a tangent to the circle.

DIRECT AND TRANSVERSE COMMON TANGENTS

When a line is drawn which touches each of two given circles the line is called

(i) *a direct common tangent* to the two circles if the two points of contact of the tangent with the circles are on the *same* side of the line joining the centres of the circles, and

common tangent if the two points of contact
ite sides of the line of centres.

CONSTRUCTION 14

To construct two direct common tangents to two given unequal circles.

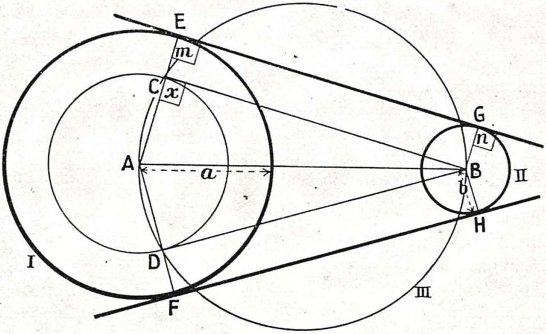


FIG. 347

Given circles I and II, centres A and B, radii a and b ($a > b$).

Construct two direct common tangents to the circles I and II.

Construction. Draw circle III on AB as diameter.

Draw a circle, centre A, radius ($a - b$), to cut circle III at C and D.

Join AC and AD, and produce to meet circle I at E and F.

Draw BG and BH parallel to AE and AF respectively.

Join EG and FH. These are the required tangents.

Proof. We know that CE = AE − AC;

i.e., CE $= a - (a - b) = b$.

∴ CE = BG,

and they were drawn parallel.

∴ EG ∥ CB.

Since $x = 90°$ (angle in semicircle),

∴ $m = 90°$ (corr. angles; EG ∥ CB).

∴ EG touches the circle I at E (tangent, radius theorem).

Since $n = 90°$ (corr. angle to m; AE ∥ GB).

∴ EG touches the circle II at G (tangent, radius theorem).

∴ EG is a direct common tangent to the circles I and II.

Also FH is a direct common tangent to circles I and II.

Construction 15

To construct a transverse common tangent to two given circles.

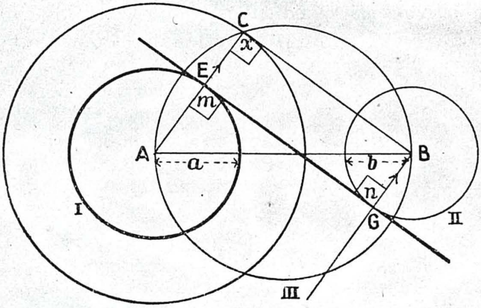


Fig. 348

Given two circles I and II, centres A and B and radii a and b.

Construct a transverse common tangent.

Construction. On AB as diameter draw a circle, III.

With centre A and radius $(a + b)$ draw a circle cutting III at C. Join AC, to cut I at E.

Draw BG parallel to CE, to cut II at G. Join EG.

Then EG is a required tangent.

Proof. We know that AC $= a + b$ (construction),

and AE $= a$ (given).

$\therefore$ EC $= b$,

i.e., EC = GB

Also EC ∥ GB (construction).

$\therefore$ CB ∥ EG.

Since $x = 90°$ (angle in semicircle),

$\therefore$ $m = 90°$ (corr. angles; CB ∥ EG).

$\therefore$ EG touches circle I at E (tangent, radius theorem).

Since $n = 90°$ (alt. $\angle$ to m; AC ∥ GB),

$\therefore$ EG touches circle II at G (tangent, radius theorem).

Hence EG is a transverse tangent to circles I and II.

EXERCISES: XLVII

1. Draw a circle with centre S and radius 4 cm. Take a point T 7 cm. from S. From T draw two tangents to the circle.

2. Draw a straight line ST 7 cm. long. Through T draw a straight line whose perpendicular distance from S is 4 cm.

3. Draw two circles, radii 2·3 in. and 0·8 in., with their centres 3·9 in. apart. Draw their direct common tangent.

4. Calculate the length of the direct common tangent in the previous exercise.

5. Draw two circles, radii 2 in. and 1·6 in., with their centres 3·9 in. apart. Draw their transverse common tangent.

6. Calculate the length of the transverse common tangent in the previous exercise.

7. Draw a straight line AB 3·9 in. long. Draw a straight line to cut AB produced so that the perpendiculars from A and B on to this line are 2·3 in. and 0·8 in. long.

8. Draw a straight line AB 3·9 in. long. Draw a straight line to cut AB internally so that the perpendiculars from A and B on to this line are 2 in. and 1·6 in. long.

9. Circles, radii a in. and b in., are drawn with their centres c in. apart. The direct common tangent is x in. long and the transverse common tangent is y in. long. Prove that

(i) $x^2 + a^2 + b^2 - 2ab = c^2$.
(ii) $y^2 + a^2 + b^2 + 2ab = c^2$.

10. Draw two equal circles, radius 1 in., with their centres 3 in. apart. Draw a direct common tangent.

11. The centres of two circles are 5·2 in. apart. The lengths of their direct and transverse common tangents are 4·8 in. and 2 in. Calculate their radii.

12. Draw a circle of radius 5 cm. Draw a second circle which is such that each tangent to it cuts off from the first circle a chord 6 cm. long.

13. Draw a circle with centre A and radius 5 cm. Find a point 9 cm. from A. Through this point draw a line which cuts off from the circle a chord 6 cm. long. (Cf. Example on p. 265.)

14. Show how to construct a straight line which cuts off from two given circles chords of given lengths.

15. Draw a circle of radius 2 in. Draw two tangents to this circle so that the angle between the tangents is 70°.

16. Draw the locus of the point of intersection of the two tangents in the previous exercise.

17. Draw a circle with centre A and radius 4 cm. Take a point B 2 cm. from A. Draw two tangents of length 7 cm. to the circle so that their chord of contact passes through B.

18. Show how to construct through one of the points of intersection of two circles a chord of one circle which is bisected by the circumference of the second.

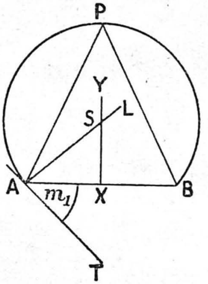
19. At a point A on a circle of radius 4 cm. draw a tangent AT. Find a point B on AT such that the circle cuts off from a line drawn through B making an angle of 40° with AT a chord 5 cm. long.

20. On a given line find a point such that the tangents from this point to a given circle contain a given angle.

21. Draw a quadrilateral ABCD with DC = 8 cm., DA = 2 cm., BC = 2·5 cm., $\widehat{D} = 70°$, and $\widehat{C} = 80°$. Draw a circle to touch DC, DA, and AB. Draw a second circle to touch DC, CB, and AB. Find points P and Q on AB and DC such that the line PQ touches both circles. (Notice that you have divided a given quadrilateral into two quadrilaterals each of which has an inscribed circle.) If particular numbers are not given, what is the condition such that the line PQ can be drawn?

Construction 16

To construct the locus of a variable point at which a given fixed straight line subtends a given angle, if the moving point lies on one side only of the given straight line.



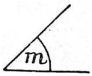


Fig. 349

Given a fixed straight line AB and an angle m.

Construct the locus of a variable point at which AB subtends an angle equal to m.

Construction. Make $\widehat{BAT} = m$. Construct $\widehat{TAL} = 90°$.

Construct XY, the right bisector of AB, to cut AL at S.

With centre S and radius SA construct a circle.

The arc of this circle which lies on the side of AB remote from T is the required locus.

Proof. We know that SA = SB (SX is rt. bisector of AB).

$\therefore$ the circle passes through B.

AT is a tangent to the circle (SA radius; $\widehat{SAT} = 90°$).

$\therefore$ $m_1 = \widehat{P}$ (tangent, chord theorem).

$\therefore$ $\widehat{P} = m$.

$\therefore$ arc APB is the required locus.

Note. In a numerical example a shorter method is obtained by making use of the fact that the angle at the centre of a circle is double the angle at the circumference standing on the same arc. The first example on p. 265 illustrates this method.

CONSTRUCTION 17

To construct in a given circle a triangle whose angles shall be equal to those of a given triangle.

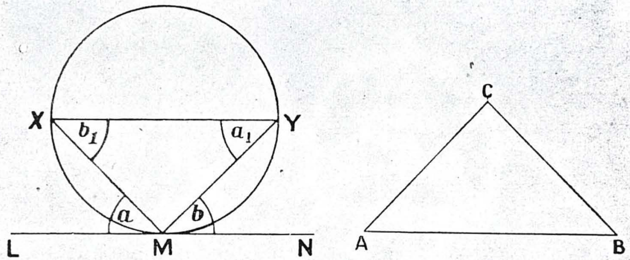


FIG. 350

Given a circle and a △ ABC.

Construct a triangle in the circle equiangular to △ ABC.

Construction. At any point M on the circle draw a tangent LMN.

Make $\widehat{LMX} = \widehat{A}$, $\widehat{NMY} = \widehat{B}$.

Then XMY is the required triangle.

Proof. Since $a = a_1$ (tangent, chord theorem),

and $a = \widehat{A}$ (construction),

$\therefore\ a_1 = \widehat{A}$.

Similarly $b_1 = \widehat{B}$.

$\therefore\ \widehat{XMY} = \widehat{C}$ (third angles of △s);

i.e., △ MXY is equiangular to △ ABC.

NOTE. A shorter method when numbers are given is obtained by using the fact that the angle at the centre of a circle is twice the angle at the circumference standing on the same arc. The second example on p. 265 illustrates this method.

Example. Draw a triangle ABC in which AB = 5 cm., $\widehat{ACB}$ = 40°, and the median CD = 6 cm.

Draw: (i) AB = 5 cm.; (ii) $a = a_1 = 50°$ (determines S); (iii) circle, centre S, radius SA; (iv) circle, centre D, radius 6 cm.

The third and fourth steps determine C.

△ ABC is the required triangle.

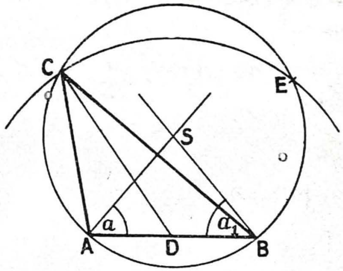


Fig. 351

Example. Draw a circle with centre S and radius $1\frac{1}{2}$ in. Take any point P distance 2 in. from S. Inscribe in the circle a △ ABC such that AB produced passes through P, $\widehat{A}$ = 40°, and $\widehat{C}$ = 75°.

Construction. Draw a circle, radius $1\frac{1}{2}$ in., centre S.

Take any radius SY.

Make $\widehat{YSZ}$ = 80°, $\widehat{YSX}$ = 150°.

Draw SL perpendicular to XY.

Draw a circle, centre S, radius SL.

Draw the circle, on SP, as diameter.

Let these circles meet at M.

Join PM, cutting the given circle at A and B.

Make $\widehat{CAB}$ = 40°.

Then ABC is the required triangle.

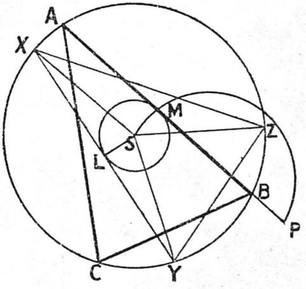


Fig. 352

Note. Since there are two positions for M, there are two solutions of the problem.

Example. Draw a circle, radius 1 in. Circumscribe the circle by a △ ABC with $\widehat{A} = 80°$ and $\widehat{B} = 45°$. Justify your construction.

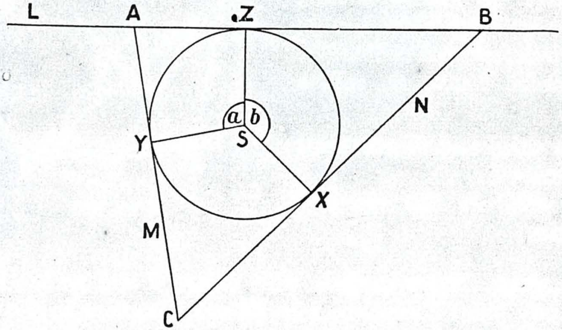


Fig. 353

Construction. Draw

(i) a circle with centre S and radius 1 in.,

(ii) any radius SZ,

(iii) $a = 100°$ and $b = 135°$,

and (iv) $\widehat{LZS} = 90°$, $\widehat{MYS} = 90°$, and $\widehat{NXS} = 90°$.

Then the intersections of LZ, MY, and XN give the points A, B, and C, and ABC is the required triangle.

Justification.

Since $\widehat{AZS} = 90°$, $a = 100°$, and $\widehat{AYS} = 90°$,

$\therefore$ $\widehat{YAZ} = 80°$ (fourth angle of quad[l]).

Similarly $\widehat{XBZ} = 45°$.

Also AB touches the circle (AB perp[r] radius SZ).

Similarly AC and BC are tangents.

$\therefore$ ABC is the required triangle.

EXERCISES: XLVIII

1. On a straight line 2 in. long draw a segment of a circle to contain an angle of 70°.

2. On a straight line 2 in. long draw a segment of a circle to contain an angle of 130°.

3. From a given circle show how to cut off a segment to contain a given angle.

4. Draw a circle of radius 1 in. Circumscribe a triangle with angles 50°, 60°, and 70°.

5. Draw a circle of radius 2 in. Inscribe a triangle with angles 55°, 75°, and 50°.

6. Draw a $\triangle$ ABC in which CD is a median, AB = 2 in., $\widehat{ACB} = 50°$, and CD = $1\frac{1}{2}$ in.

7. Draw a $\triangle$ ABC in which CD is a median, AB = 6 cm., $\widehat{ACB} = 100°$, and $\widehat{ACD} = 60°$.

8. Draw a $\triangle$ ABC with AB = 3 in., $\widehat{ACB} = 70°$, and the area of $\triangle$ ABC = 2·4 sq. in.

9. Draw a $\triangle$ ABC with sides 6 cm., 7 cm., and 8 cm. in length. Find a point X inside the $\triangle$ such that $\widehat{AXB} = \widehat{AXC} = \widehat{BXC}$.

10. Draw a circle of radius 2 in. Inscribe a $\triangle$ ABC with AB = 1·8 in. and perimeter 5 in.

11. Show how to construct a triangle given the length of one side, the angle opposite to that side, and the sum of the lengths of the other two sides.

12. BD is a median of the $\triangle$ ABC in which $\widehat{A} = 70°$ and E is the mid-point of BC. Calculate $\widehat{EDC}$. Draw a $\triangle$ ABC in which BC = 3 in., $\widehat{A} = 70°$, and the median BD = $2\frac{1}{2}$ in.

13. Given a chord AB of a circle and X a point on AB, find a point C on the circumference of the circle such that XC bisects $\widehat{ACB}$. (*Hint.* Produce CX to cut the circle at D.)

14. Draw a circle of radius 2 in. Inscribe a quadrilateral ABCD such that the tangent at A makes an angle of 40° with AB, $\widehat{BDC} = 20°$, and the tangent at D makes an angle of 30° with DC.

15. Draw a quadrilateral ABCD in which BC = 3 cm., DA = 4 cm., $\widehat{BAC} = \widehat{BDC} = 40°$, and the angle between DA and CB (both produced) is equal to 36°.

CONSTRUCTION 18

To construct a circle to touch a given circle and to touch a given straight line at a given point.

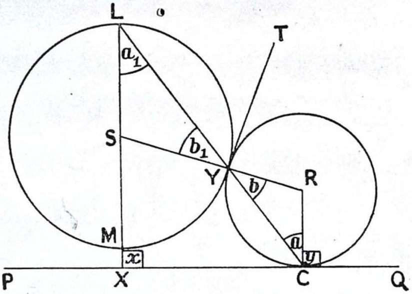


FIG. 354

Given a circle, centre S, and C a point on the straight line PQ.

Construct a circle to touch the given circle and to touch PQ at C.

Construction. Through S draw LSMX perpendicular to PQ to cut the given circle at L and M.

Join LC to cut the given circle at Y.

Let SY produced meet a line through C parallel to XL at R.

Draw a circle, centre R, radius RC.

This is the required circle.

Proof. Since $b = b_1$ (vertically opposite),

$b_1 = a_1$ (LS = SY),

and $a_1 = a$ (alt. angles; LX || RC),

$\therefore$ $b = a$.

$\therefore$ RC = RY.

$\therefore$ the circle with centre R and radius RC passes through Y.

Draw YT perpendicular to SR.

Then YT is a tangent to both circles

(being perpendicular to a radius of each).

$\therefore$ the circles touch at Y.

Also $y = x$ (corr. angles; LX ∥ RC).

∴ PQ touches the circle, centre R, at C

(PQ perp^r to radius RC).

∴ the circle, centre R, is the required circle.

NOTE. A second circle can be drawn to touch PQ at C and to touch the given circle internally. This construction is left as an exercise to the pupil. (*Hint.* Produce, if necessary, CM to cut the given circle at Z, and let ZS produced cut a parallel through C to XL.)

EXAMPLE. Describe a regular pentagon about a circle of radius $1\frac{1}{2}$ in. Justify your construction.

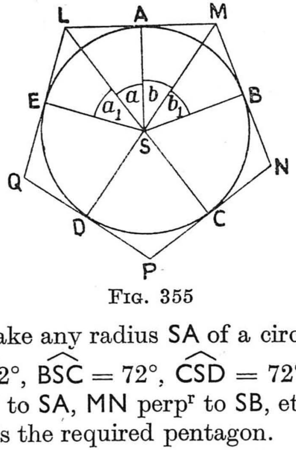


FIG. 355

Construction. Take any radius SA of a circle of radius $1\frac{1}{2}$ in.

Make $\widehat{ASB} = 72°$, $\widehat{BSC} = 72°$, $\widehat{CSD} = 72°$, and $\widehat{DSE} = 72°$.

Draw LM perp^r to SA, MN perp^r to SB, etc.

Then LMNPQ is the required pentagon.

Proof. It is left to the pupil to supply the proof by justifying the following statements:

(i) $\widehat{ASE} = 72°$. (iv) LM = LQ.

(ii) $a = 36°$. (v) $\widehat{QLM} = 108°$.

(iii) LM = 2LA. (vi) $\widehat{QLM} = \widehat{LMN}$.

EXERCISES: XLIX

1. Draw a circle of radius 2 in. Inscribe a square.

2. Inscribe in a circle of radius 6 cm. a rectangle one of whose sides is 8 cm.

3. Circumscribe a square about a circle whose radius is 1·5 in.

4. Inscribe a regular pentagon in a circle of radius 5 cm.

5. Draw a circle to touch the sides of a regular hexagon with sides 4 cm. long.

6. Show how to draw a tangent to a given circle at a given point on it without finding the centre of the circle.

7. In a given circle inscribe a right-angled isosceles triangle.

8. Inscribe in a given triangle a semicircle whose diameter lies along one side of the triangle and which touches the other two sides.

9. Construct a triangle given the base and the lengths of the perpendiculars from the ends of this base on to the opposite sides.

10. Draw a circle with centre A, radius 4 cm. Draw a line BC 5 cm. from A. Find a point P on this line such that the tangents from P to the circle contain an angle of 50°.

11. Inscribe a circle in a given sector of a given circle.

12. Given a circle and two tangents to it, construct a circle to touch the given circle and the two tangents.

13. Show how to construct through one of the points of intersection of two circles a chord of the first circle whose mid-point lies on the circumference of the second circle.

14. In a square with sides 5 in. long inscribe four equal circles each of which touches two of the circles and two sides of the square.

15. Draw a circle of radius 1·5 in. and circumscribe it by a rhombus one of whose angles is 50°.

16. In a circle of radius 5 cm. take a point A 2 cm. from the centre. Inscribe in the circle a $\triangle$ PQR with PQ passing through A, $\widehat{P} = 47°$, and $\widehat{R} = 71°$.

17. Draw a circle of radius 5 cm. Draw a $\triangle$ ABC with BC touching the circle, AB and AC produced touching the circle, $\widehat{B} = 53°$, and $\widehat{C} = 42°$.

18. Show how to inscribe in a given circle a triangle with given angles, one of whose sides (produced if necessary) touches a second given circle.

19. Show how to inscribe in a given circle a triangle two of whose sides shall pass through two given points, and whose third side shall be a given distance from the centre.

20. Show how to describe a triangle given the perimeter, the angle at one vertex, and the length of the altitude through that vertex.

21. Construct a right-angled triangle, given the length of the hypotenuse and the length of the altitude on to the hypotenuse.

22. Show how to circumscribe a rhombus with given side about a given circle.

23. In a circle of radius 5 cm. draw a chord AB 7 cm. long and a chord CD 6 cm. long whose mid-point lies on AB.

24. A square is described in a given quadrant of a circle, having two vertices on the circular arc and one vertex on each bounding radius. Given that the side of the square is $2x$ in. and the radius of the circle y in., prove that $10x^2 = y^2$. Hence show how to inscribe such a square in a given quadrant.

25. In a $\triangle$ ABC the altitude AD = 3 in., $\widehat{BAC} = 70°$, and MA = 3·5 in., where M is on BC and AM bisects $\widehat{BAC}$. Draw the triangle.

26. Circumscribe a square about a given quadrilateral.

27. Prove that if I is the in-centre of $\triangle$ ABC, then

$$\widehat{AIB} = 90° + \tfrac{1}{2}\widehat{C}.$$

Hence show how to construct a triangle given the base, the radius of the inscribed circle, and the vertical angle of the triangle.

28. Show how to construct a triangle given the radius of the in-circle, the vertical angle, and the perimeter.

29. Given the centres of the three escribed circles of a triangle, show how to construct the triangle.

30. Show how to construct a triangle given the centres and radii of the inscribed circle and one escribed circle.

TEST PAPERS

PAPER LV

1. AB is a side of a regular octagon inscribed in a circle, and AC is a side of a regular pentagon inscribed in the same circle. Find $\widehat{BAC}$ when $\widehat{BAC}$ is obtuse.

2. Prove that all circles which pass through a fixed point and have their centres on a given straight line pass through a second fixed point.

3. Two circles with centres at X and Y touch at A, and BC is a common tangent touching the circles at B and C. Prove that XY touches the circle BAC.

PAPER LVI

1. The chords AB and CD of a circle, centre S, are perpendicular. Given that $\widehat{ASC} = 70°$, find $\widehat{BSD}$.

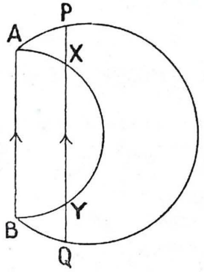


FIG. 356

2. Given that in Fig. 356 PQ is parallel to AB, prove that PX = YQ.

3. Circumscribe a rhombus about a given circle.

PAPER LVII

1. Find the radius of the circle inscribed in a quadrant of a circle whose radius is 6 in.

2. In the cyclic pentagon ABCDE, AE = DC and AB = ED. Prove that BE = CE.

3. Construct a triangle given one side and the lengths of the two altitudes drawn through the ends of that side.

PAPER LVIII

1. Two parallel chords, 5 in. and 12 in. long, are drawn in a circle of diameter 13 in. Find the distance between the chords.

2. In the $\triangle$ ABC, AB = AC. A circle through B and C cuts AB at D and AC at E. S is the centre of the circle ADE. Prove that the angles ASD and BAC are supplementary.

3. Draw any $\triangle$ ABC. By a geometric construction find the in-centre I and the ex-centres I_1, I_2, and I_3. Measure with the protractor the angle between II_1 and I_2I_3.

PAPER LIX

1. A variable chord PQ is drawn through the mid-point M of a fixed chord AB of a given circle. Given that the radius of the circle is 6 cm. and AB is 4 cm. long, find the greatest and least lengths of PQ.

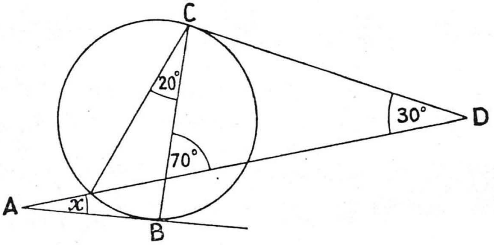


FIG. 357

2. In Fig. 357 AB and CD are tangents. Find the value of x. Also prove that the two chords through C are equal.

3. Prove that the right bisector of one side of a given triangle meets the bisector of the opposite angle at a point on the circum-circle of the triangle.

PAPER LX

1. Draw two circles, radii 3 cm. and 5·1 cm., whose centres are 6·3 cm. apart. Measure the length of their common chord. Check by calculation.

2. In the cyclic quadrilateral ABCD, AD = DC and BD = BC. Given that $\widehat{DBC} = 40°$, find which is greater, AB or DC.

3. In the $\triangle$ ABC, AB = AC. A circle is drawn through C and A touching AB at A and cutting BC at D. Prove that $\triangle$ ABD is isosceles.

Paper LXI

1. Draw the circum-circle of the $\triangle$ ABC in which AB = 5·5 cm., BC = 6·2 cm., and CA = 6·7 cm., and measure its radius.

2. CD is the diameter of the circle ABC, and the perpendicular from C to AB is produced to cut the circum-circle at E. Prove that the minor arcs AE and BD are equal.

3. Two fixed circles ABP and ACQ touch internally at A. The diameter AC of the outer circle cuts the inner circle at B. Q is a variable point on the outer circle, and AQ cuts the inner circle at P. D is the mid-point of BC, and R is the mid-point of PQ. Prove that $\widehat{ARD} = 90°$. What is the locus of R?

Paper LXII

1. Two concentric circles have radii of 13 cm. and 5 cm. Find the length of a chord of the larger circle which touches the inner circle.

2. Two points A and B are taken on a circle with centre S, and AS is produced to a point C. Prove that AC > CB.

3. A point D is taken on the side BC of a $\triangle$ ABC such that $\widehat{ADB} = \widehat{BAC}$. Prove that BA touches the circle ADC.

Paper LXIII

1. The points A and B are 4 cm. apart. Draw a circle of radius 5 cm. to pass through A and B.

2. The quadrilateral ABCD is cyclic. The bisectors of angles BAC and BDC meet at X. Prove that X lies on the circle ABCD.

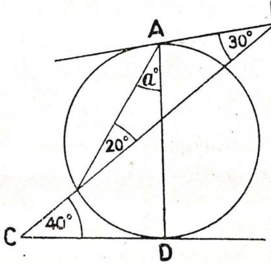


Fig. 358

3. In Fig. 358 AB and CD are tangents. Find the value of a.

PAPER LXIV

1. From a point 8·5 cm. from the centre of a circle of radius 4 cm. two tangents are drawn. Find the distance of their chord of contact from the centre of the circle.

2. A point C is taken on the circumference of a circle with diameter AB. The bisector of $\widehat{ABC}$ meets AC at E and meets at D the tangent at A. Prove that $\triangle$ ADE is isosceles.

3. Draw a $\triangle$ ABC with AB = 3 in., BC = 2·6 in., and CA = 2 in. Find a point O inside the triangle such that $\widehat{AOB} = 100°$ and $\widehat{BOC} = 120°$.

PAPER LXV

1. Draw a $\triangle$ ABC with AB = 2 in., BC = 3 in., and CA = $2\frac{1}{2}$ in. Draw a circle to pass through A and to touch BC at C.

2. Prove that if two circles cut orthogonally (that is, the tangents to the circles at a common point are perpendicular), then the tangent to one circle at one of the points of intersection of the circles passes through the centre of the other circle.

3. A point D is taken inside the $\triangle$ ABC, E is any point on the circle ADB (E and D being on opposite sides of AB), EA is produced to cut the circle ADC at X, and EB is produced to cut the circle BDC at Y. Prove that XCY is a straight line.

PAPER LXVI

1. Show how to construct a right-angled triangle given its hypotenuse and its area.

2. The diameters AOB, COD, of a given circle are perpendicular, X is any point on the minor arc AC, XE is the perpendicular on to AB, and OX meets the circle on OC as diameter at P. Prove that OP = XE.

3. Two circles cut orthogonally. Their radii are 12 in. and 5 in. Find the length of their common chord.

PAPER LXVII

1. Draw a $\triangle$ ABC with BC = 2 in., $\widehat{BAC} = 65°$, and the median AD = 1·4 in.

2. The diagonals of the quadrilateral ABCD meet at O. Prove that the centres of the circles AOB, BOC, COD, and DOA lie at the vertices of a parallelogram.

3. A point P moves on a fixed circle, O is a fixed point within the circle, and OP is produced to Q making PQ = OP. What is the locus of Q?

Paper LXVIII

1. Straight lines CPX and CQY are drawn through the mid-point of the arc AB of a circle, cutting the chord AB at P and Q and the circle at X and Y. Prove that PQYX is a cyclic quadrilateral.

2. In the △ ABC, $\widehat{ABC} = 40°$, S is a point inside the triangle on the right bisector of AB, and D is a point on BC such that SD = SB. Find the value of $\widehat{ASD}$.

3. Three circles with centres at A, B, and C touch externally at X, Y, and Z. Prove that circle XYZ is the inscribed circle of △ ABC.

Paper LXIX

1. Each of three equal circles, radius 2 in., touches the other two circles externally. Find the radius of the circle which touches each of the three circles internally.

2. In the △ ABC, AB = AC, the circle on AB as diameter cuts AC at D, and a point E is taken on the perpendicular through A to AB. Given that E and C are on the same side of AB and AE = BD, prove that EC is parallel to AB. (*Hint.* Show that △s ABC, ABE, are equal in area.)

3. In the △ ABC, AB = BC, AB is produced to D, and the bisectors of angles CAB and CBD meet at X. If AB is fixed and C is variable, find the locus of X.

Paper LXX

1. In the △ ABC, AB = BC and $\widehat{ABC} = 80°$. A circle is drawn passing through B and C and touching AB at B. Prove that if the circle cuts AC at X, then CX is equal to the radius of the circle.

2. Draw the quadrilateral ABCD with AB = 0·5 in., BC = 1·7 in., CD = 1·9 in., AD = 2·5 in., and $\widehat{C} = 90°$. Draw a circle through A, D, and C. Verify that this circle passes through B. Prove that the circle ADC passes through B.

3. Two circles touch externally at A. A third circle touches the other two internally at B and C. Prove that the tangents at A, B, and C meet in a point.

PAPER LXXI

1. The fixed chord AB of a circle is 10 cm. long. PQ is any non-intersecting diameter of the circle and is 26 cm. long. Find the sum of the lengths of the perpendiculars from P and Q on to AB.

2. The side BC of the $\triangle$ ABC is produced to D, and the bisectors of angles ABC and ACD meet at X. Prove that if S is the centre of the circle BXC, then S lies on the circle ABC.

3. Draw a circle with centre A and radius 4 cm. Draw a chord of the circle, length 6 cm., to pass through a given point B, 3 cm. from A.

PAPER LXXII

1. Draw the $\triangle$ ABC given AB $=$ 4 in., $\widehat{ACB} = 70°$, and the point D in which the bisector of $\widehat{ACB}$ meets AB.

2. Fixed points A and B are taken on a given circle. C and D are variable points on the circle such that AC is parallel to BD. Prove that CD always touches a fixed concentric circle.

3. In the $\triangle$ ABC, BC $= a$ in., CA $= b$ in., AB $= c$ in., the radius of the ex-circle opposite to A $= x$ in., and the area of $\triangle$ ABC $= \Delta$ sq. in. Prove that $2\Delta = x(b + c - a)$.

PAPER LXXIII

1. Draw a cyclic quadrilateral ABCD with DC $=$ 3 in., AB $=$ 2 in., $\widehat{DBC} = 30°$, and $\widehat{ACD} = 40°$.

2. In the quadrilateral ABCD, $\widehat{BAD} = 90°$, AB $=$ 2DC, DC $=$ 2 AD, and $BC^2 = 13AD^2$. Prove that $\widehat{DAC} = \widehat{DBC}$.

3. The diagonals of the parallelogram ABCD meet at X. Given that AB is fixed and D moves on a fixed circle through A and B, prove that X moves on a fixed circle passing through B and the mid-point of AB.

PAPER LXXIV

1. Three equal spheres, radius 1 ft., rest on the horizontal floor touching one another. A fourth sphere, radius 8 in., rests on the other three. Find the height of its centre above the floor.

2. Two circles with centres at A and B and radii a and b touch externally, and PQ is a direct common tangent. Given that M is the mid-point of AB and MX is perpendicular to PQ, prove that

MX $= \frac{1}{2}(a + b)$. Hence prove that the circle on AB as diameter touches PQ.

3. Prove that the tangents at the extremities of two perpendicular chords of a circle form a cyclic quadrilateral.

PAPER LXXV

1. I is the in-centre of $\triangle$ ABC and AI is produced to cut the circle ABC at S. Prove that S is the centre of $\triangle$ BIC.

2. Two circles cut at X and Y. A variable point P moves on one circle, and PX and PY are produced to cut the second circle at A and B. Prove that AB touches a fixed circle.

3. Given that A and B are the centres of two fixed non-intersecting circles and P is a variable point such that the tangents from P to the two circles are equal, find the locus of P. (*Hint.* Draw PM perpendicular to AB and prove that $PA^2 \sim PB^2 =$ constant.)

PAPER LXXVI

1. Show how to draw a straight line which will cut off from two given circles chords of given lengths.

2. Two circles touch at A. The tangent at B to one circle meets the tangent at C to the other circle at X. Prove that $\widehat{BXC} + 2\widehat{BAC} =$ 4 rt. angles.

3. In the $\triangle$ ABC, AB is fixed and C moves so that $\widehat{ACB} = 90°$. The medians of the triangle meet at G. Find the locus of G. (*Hint.* Use Exercises: XX, No. 22.)

PART V

FURTHER AREAS AND RATIO

ALGEBRAIC IDENTITIES

A. *To illustrate geometrically the algebraic identity*

$$k(a+b+c) \equiv ka+kb+kc.$$

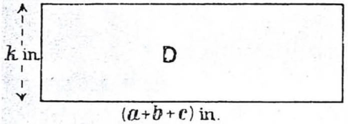


FIG. 359

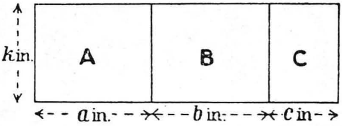


FIG. 360

Draw rectangles as shown in Figs. 359 and 360.

We see that area $D =$ sum of areas A, B, and C,

$$i.e., \quad D = A + B + C.$$

$$\text{But} \quad D = k(a+b+c) \text{ sq. in.,}$$

$$\text{and} \quad A + B + C = (ka+kb+kc) \text{ sq. in.}$$

$$\text{Hence} \quad k(a+b+c) \equiv ka+kb+kc.$$

The formal theorem in geometry corresponding to this identity is:

If two straight lines be taken, one of which is divided into any number of parts, then the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the separate parts of the divided line.

B. *To illustrate geometrically the algebraic identity*

$$(a + b)^2 \equiv a^2 + 2ab + b^2.$$

Divide the square shown in Fig. 361 into the square and rectangles as in Fig. 362.

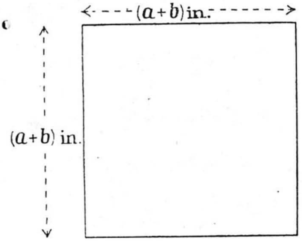


FIG. 361

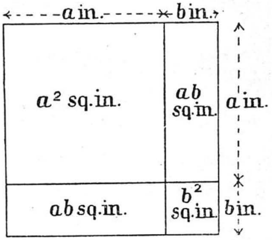


FIG. 362

The area of Fig. 361 = the area of Fig. 362.

Area of Fig. 361 $= (a + b)^2$ sq. in.,

and area of Fig. 362 $= (a^2 + ab + ab + b^2)$ sq. in.

Thus $(a + b)^2 \equiv a^2 + 2ab + b^2$.

The theorem in geometry to which this identity corresponds is:

If a straight line be divided into two parts the square on the whole line is equal to the sum of the squares on the separate parts together with twice the rectangle contained by these parts.

C. *To illustrate geometrically the algebraic identity*

$$(a - b)^2 \equiv a^2 - 2ab + b^2.$$

Divide the sum of the squares in Fig. 363 into the square and rectangles shown in Fig. 364.

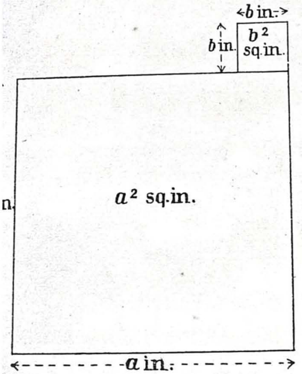


FIG. 363

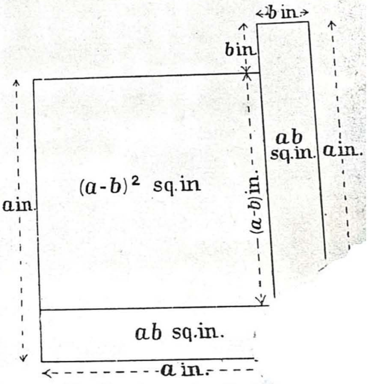


FIG. 364

The area of Fig. 363 = the area of Fig. 364.

But area of Fig. 363 $= (a^2 + b^2)$ sq. in.,

and area of Fig. 364 $= \{(a - b)^2 + ab + ab\}$ sq. in.

Thus $a^2 + b^2 \equiv (a - b)^2 + 2ab.$

$\therefore\ a^2 - 2ab + b^2 \equiv (a - b)^2.$

The theorem in geometry to which this id‹ sponds is:

If a straight line be divided into two par‹
the first part is equal to the sum ‹
the whole line and on the second ‹
twice the rectangle contained b‹
the second part.

D. *To illustrate geometrically the algebraic identity*

$$a^2 - b^2 \equiv (a + b)(a - b).$$

Divide the square in Fig. 365 into a square and rectangles as shown.

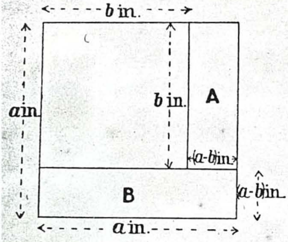


FIG. 365

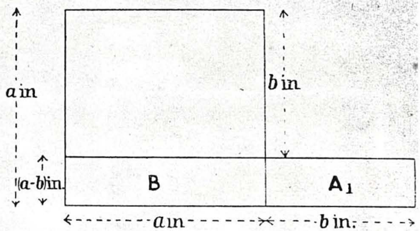


FIG. 366

Fig. 366 A_1 is a new position of A.

. 365 the area of $(A + B) = (a^2 - b^2)$ sq. in.,

ig. 366 the area of $(A_1 + B) = (a + b)(a - b)$ sq. in.

Thus $a^2 - b^2 \equiv (a + b)(a - b)$.

theorem in geometry corresponding to this identity is:

ıe difference of the squares on two given lines is equal to the rectangle contained by their sum and difference.

PLE. If A, P, B, and C are points in order on a straight hat $BC = 2AB$, then

$$PC^2 + 2AP^2 = 3(BP^2 + 2AB^2).$$

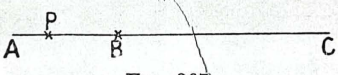


FIG. 367

line APBC with $BC = 2AB$.

$AP^2 = 3(BP^2 + 2AB^2)$.

Proof. Let $AP = x$ in. and $AB = a$ in.

Then $BC = 2a$ in.

$$PC^2 + 2AP^2 = \{(3a - x)^2 + 2x^2\} \text{ sq. in.}$$
$$= (9a^2 - 6ax + x^2 + 2x^2) \text{ sq. in.}$$
$$= (9a^2 - 6ax + 3x^2) \text{ sq. in.}$$

Also
$$3(BP^2 + 2AB^2) = 3[(a - x)^2 + 2a^2] \text{ sq. in.}$$
$$= 3[a^2 - 2ax + x^2 + 2a^2] \text{ sq. in.}$$
$$= (9a^2 - 6ax + 3x^2) \text{ sq. in.}$$

$$\therefore \quad PC^2 + 2AP^2 = 3[BP^2 + 2AB^2]. \qquad \textit{Q.E.D.}$$

Exercises: L

1. Illustrate geometrically the following algebraic identities:

(i) $x(y - z) \equiv xy - xz$.
(ii) $(a + b)(c + d) \equiv ac + ad + bc + bd$.
(iii) $(x - y)^2 + (x + y)^2 \equiv 2x^2 + 2y^2$.
(iv) $(x + y)^2 - (x - y)^2 \equiv 4xy$.

2. A, B, C, and D are four points taken in order on a straight line, and $AB = BC$. Show that:

(i) $4AB.BD = AD^2 - CD^2$.
(ii) $2AB^2 = AB.AD - BC.CD$.
(iii) $AD^2 + CD^2 = 2BD^2 + 2AB^2$.

3. Given that ABCD is a straight line, prove that

$$AC^2 + BD^2 + 2AB.CD = AD^2 + BC^2.$$

4. If ABLM is a straight line and C and N bisect AB and LM respectively, then $2CN.AB = AL.AM - BL.BM$. Prove this proposition.

5. Prove that if ABCD is a straight line with $BC^2 = 2AB.CD$, then $AC^2 + BD^2 = AD^2$.

6. ABCD is a straight line, and $BD^2 = 4AB^2 + CD^2 + 2BC.CD$. Prove that $BC = 2AB$.

7. If ABC is a straight line and $AC.BC = AB^2$, prove that

(i) $AC^2 = AB^2 + AB.AC$.
(ii) $(AB - CB)^2 = 3CB^2 - AB^2$.

PROJECTION

Lines AB and XY are drawn, and from A and B, AL and BM are drawn to XY perpendicular to XY. LM is said to be the *projection* of AB on XY (Fig. 368).

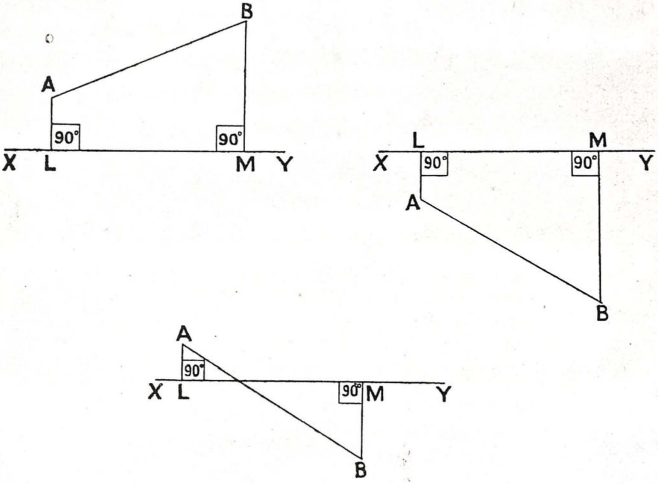


FIG. 368

If A happens to lie on the line XY (Fig. 369) the perpendicular drawn from A to XY cuts XY at A. The projection of

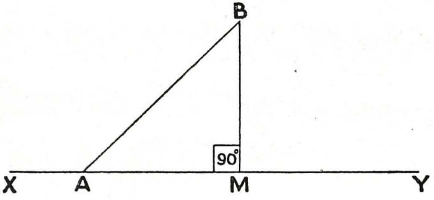


FIG. 369

AB on the line XY is now AM. Similarly, the projection of AB on the line MB is MB.

THEOREM 48: EXTENSION OF PYTHAGORAS' THEOREM

In any triangle the square on the side opposite to an *acute* angle is equal to the sum of the squares on the other two sides *minus* twice the rectangle contained by one of those sides and the projection of the other upon it.

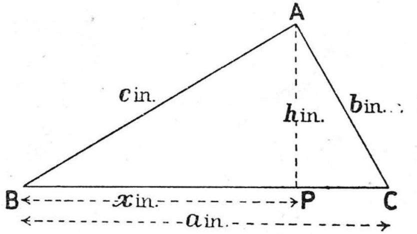


FIG. 370

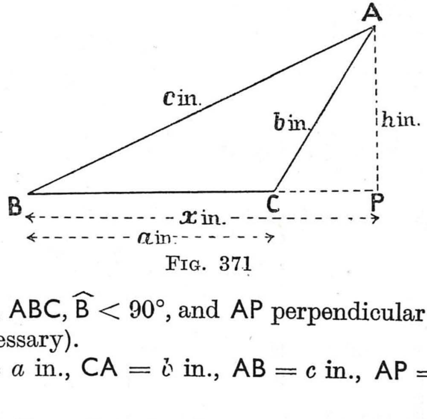


FIG. 371

Given a △ ABC, $\widehat{B} < 90°$, and AP perpendicular to BC (produced if necessary).

Let BC $= a$ in., CA $= b$ in., AB $= c$ in., AP $= h$ in., and BP $= x$ in.

Prove that $b^2 = a^2 + c^2 - 2ax$.

Proof. In Fig. 370 $b^2 = h^2 + (a - x)^2$ $(\widehat{APC} = 90°)$.

In Fig. 371 $b^2 = h^2 + (x - a)^2$ $(\widehat{APC} = 90°)$.

$\therefore$ in both figures, $b^2 = (h^2 + x^2) + a^2 - 2ax$.

Since $c^2 = h^2 + x^2$ $(\widehat{APB} = 90°)$,

$\therefore$ $b^2 = c^2 + a^2 - 2ax$. *Q.E.D.*

THEOREM 49

In any obtuse-angled triangle the square on the side opposite to an *obtuse* angle is equal to the sum of the squares on the other two sides *plus* twice the rectangle contained by one of those sides and the projection of the other one upon it.

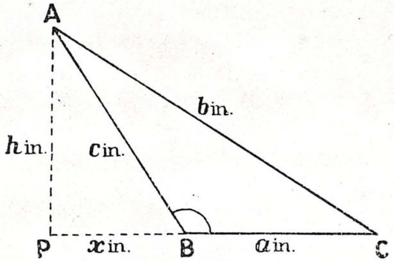


FIG. 372

Given a △ ABC, $\widehat{B} > 90°$, and AP drawn perpendicular to CB produced.

Let BC $= a$ in., CA $= b$ in., AB $= c$ in., AP $= h$ in., and BP $= x$ in.

Prove that $b^2 = c^2 + a^2 + 2ax$.

Proof. We know that

$$b^2 = h^2 + (x + a)^2 \quad (\widehat{APC} = 90°).$$

$$\therefore \quad b^2 = (h^2 + x^2) + a^2 + 2ax.$$

Since $c^2 = h^2 + x^2 \quad (\widehat{APB} = 90°)$,

$$\therefore \quad b^2 = c^2 + a^2 + 2ax.$$

Q.E.D.

Exercises: LI

Section A

1. By calculation determine whether the $\triangle$ ABC is acute-angled, right-angled, or obtuse-angled in the following cases:

(i) AB = 5 in., BC = 6 in., CA = 7 in.
(ii) AB = 7 in., BC = 12 in., CA = 13 in.
(iii) AB = 12 in., BC = 20 in., CA = 30 in.
(iv) AB = 24 in., BC = 25 in., CA = 7 in.

2. Calculate the projection of CA on to CB in each of the triangles in the preceding exercise.

3. In $\triangle$ ABC, BC = 4 in., BA = 5 in., and $\widehat{B} = 60°$. Calculate AC^2.

4. In $\triangle$ ABC, BC = 4 in., BA = 5 in., $\widehat{B} = 120°$. Calculate AC^2.

5. The sides of a triangle are 7 in., 5 in., and 3 in. in length. Prove that one angle measures 120°.

6. Determine whether the circum-centre lies inside or outside the triangle whose sides are 7 in., 9 in., and 12 in. in length.

7. In the rectangle ABCD, AD = 18 in. and AB = 4 in. The points L and M trisect BC, and P is any point lying between L and M. Prove that $\widehat{APD}$ is obtuse.

8. Perpendiculars AD and BE are drawn from A and B on to the sides BC and AC of a $\triangle$ ABC. Prove that

$$AC.CE = BC.CD.$$

9. The altitudes AD, BE, and CF of a $\triangle$ ABC meet at H. Prove that AH.HD = BH.HE = CH.HF.

10. In $\triangle$ ABC, AD and CF are altitudes and $\widehat{A}$ is acute. Prove that if $\widehat{C}$ is also acute, then $AC^2 = BC.CD + AB.AF$.

11. What does the result of the previous exercise become when $\widehat{C}$ is obtuse?

12. Given that in $\triangle$ ABC, $\widehat{A} = 60°$, prove that

$$BC^2 = AB^2 + AC^2 - AB.AC.$$

13. What does the result of the previous exercise become when $\widehat{A} = 120°$?

14. Prove that if, in the $\triangle$ ABC, AB = 2AC and $\widehat{A} = 120°$, then $BC^2 = 7AC^2$.

15. In the $\triangle$ ABC, AB $=$ AC and CD is an altitude. Prove t. $BC^2 = 2AB.BD$.

16. The diagonals of the quadrilateral ABCD meet at right angles at E. Prove that

$$AB^2 + AD^2 + 4AC.CE = BC^2 + CD^2 + 2AC^2.$$

Section B

17. In the trapezium ABCD the sides AD and BC are parallel. Prove that $AC^2 + BD^2 = AB^2 + CD^2 + 2AD.BC$.

18. In the $\triangle$ ABC, $\widehat{A} = 45°$. Prove that the area of the triangle equals $AB^2 + AC^2 - BC^2$.

19. AB, AC, and AD are edges of a rectangular box. Prove that $\triangle$ BCD is acute-angled.

20. Calculate the projection of BD on to AD in Fig. 373.

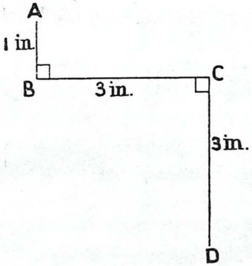


FIG. 373

21. Perpendiculars BY and CZ are drawn on to the bisector of $\widehat{BAC}$ of the $\triangle$ ABC. Prove that $BC^2 = (AB - AC)^2 + 4BY.CZ$.

22. AD is the diameter of the circle ABCD (where the points are taken in this order), and AC meets BD at E. Prove that the area of the square circumscribed to the circle equals AC.AE $+$ DB.DE.

23. Prove that if the sum of the squares on two opposite sides of a quadrilateral equals the sum of the squares on the other two opposite sides, then the diagonals of the quadrilateral are perpendicular.

THEOREM 50: THEOREM OF APOLLONIUS

In any triangle the sum of the squares on two sides is equal to twice the square on half the third side plus twice the square on the median bisecting this third side.

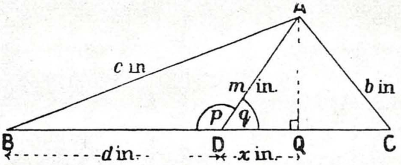


FIG. 374

Given a $\triangle$ ABC with AD a median.

Let AB $= c$ in., etc., as in Fig. 374.

Prove that $b^2 + c^2 = 2d^2 + 2m^2$.

Construction. Draw AQ perpendicular to BC.

Proof. If D and Q coincide the proof follows immediately from Pythagoras' Theorem.

If D and Q do not coincide, then either p or q is obtuse.

Suppose p is obtuse.

Then $c^2 = d^2 + m^2 + 2dx$ ($\triangle$ ABD, p obtuse),

and $b^2 = d^2 + m^2 - 2dx$ ($\triangle$ ADC, q acute).

Adding, we get

$$b^2 + c^2 = 2d^2 + 2m^2. \qquad Q.E.D.$$

NOTE. The theorem is true however small AQ may be. When A coincides with Q, Fig. 374 becomes Fig. 375.

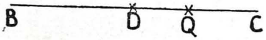


FIG. 375

If BD $=$ DC, and Q is any point on BC (produced if necessary in either direction), then

$$QB^2 + QC^2 = 2BD^2 + 2QD^2.$$

EXERCISES: LII

Section A

1. The sides of a triangle are 6 in., 7 in., and 8 in. in length. Calculate the area of the square on the median bisecting the longest side.

2. Two sides of a triangle are 8 in. and 14 in. long, and the median bisecting the third side is 7 in. in length. Calculate the length of the third side.

3. Calculate the length of the longest median of a triangle whose sides are 4 in., 7 in., and 8 in. long.

4. Two sides of a parallelogram are 4 in. and 7 in. long, and one diagonal is 9 in. long. Calculate the length of the second diagonal.

5. The diagonals of a parallelogram are 8 in. and 10 in. long, and one side is 3 in. long. Calculate the length of the other side.

6. Two sides of a triangle, of length 4 in. and 6 in., contain an angle of 120°. Calculate the length of the median bisecting the third side.

7. Repeat the previous exercise when the angle of 120° is replaced by one of 135°.

8. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

9. Given that ABCD is a rectangle and P any point, prove that $PA^2 + PC^2 = PB^2 + PD^2$.

10. C is any point on a circle with fixed diameter AB. P and Q are fixed points on AB such that AP = QB. Prove that $CP^2 + CQ^2$ is constant.

11. If A and B are fixed points and P moves so that $PA^2 + PB^2 =$ constant, what is the locus of P? What is the locus of P if the restriction that P, A, and B lie in one plane is removed?

12. Given that L and M trisect the side ALMB of a △ ABC, prove that $CA^2 - CB^2 = 3(CL^2 - CM^2)$.

13. L and M are the mid-points of the diagonals of the quadrilateral ABCD. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4LM^2.$$

Section B

14. A point C is taken on a circle with centre S and diameter AB, and CS is produced to D. Prove that

$$DA^2 + DB^2 + 4DS.SC = 2CD^2.$$

15. Prove that four times the sum of the squares on the three medians of a triangle is equal to three times the sum of the squares on the sides.

16. Prove that if two medians of a triangle are equal the triangle is isosceles.

17. State and prove the converse of Exercise 8.

18. Prove that twice the square on a median of a triangle is equal to the sum of the squares on the three sides minus half the square on the side which is bisected by the median. Deduce that the median which bisects the longest side is the shortest median.

19. Prove that, if A, C, and B are three fixed points in a straight line and P moves so that $PA^2 + AC^2 = PB^2 + BC^2$, then the locus of the mid-point of CP is the perpendicular bisector of AB.

20. Given that ABCD is a fixed parallelogram and P moves so that $PA^2 + PB^2 + PC^2 + PD^2$ is constant, find the locus of P. What is the locus of P if the restriction that P lies in the plane of the parallelogram is removed?

21. Given that A and B are fixed points and P moves on a given straight line, show how to find the position of P when $PA^2 + PB^2$ is least.

22. The mid-points of the sides BC, CA, and AB of a △ ABC are L, M, and N, and P is any point. Prove that

$$PA^2 + PB^2 + PC^2 = PL^2 + PM^2 + PN^2 + \tfrac{1}{3}(AL^2 + BM^2 + CN^2).$$

23. Points L and M are taken on the side BC of a △ ABC so that BL = LM = MC. Prove that $9AM^2 + 2BC^2 = 3AB^2 + 6AC^2$.

24. Prove that if, in the triangle ABC, $5AB^2 = AC^2 + CB^2$, then the medians through A and B are perpendicular to each other.

25. In a △ ABC the base BC is divided at X so that $m BX = n XC$. Prove that

$$m AB^2 + n AC^2 = m BX^2 + n XC^2 + (m + n) AX^2.$$

INTERSECTING CHORDS OF A CIRCLE

THEOREM 51

If two chords of a circle intersect within the circle the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other chord; each rectangle is equal to the square on the radius diminished by the square on the line joining the point of intersection of the chords to the centre of the circle.

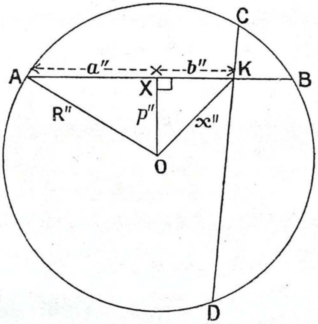


FIG. 376

Given two chords AB and CD intersecting at K in a circle with centre O and radius R in.

Prove that $AK.KB = CK.KD = R^2 - OK^2$.

Construction. Draw OX perpendicular to AB.

Proof. Let $AX = a$ in., etc., as in Fig. 376.

We know that $AX = XB$ (perp^r from centre to chord).

Since $AK = (a + b)$ in.,

and $KB = (a - b)$ in.,

$\therefore\ AK.KB = (a + b)(a - b)$ sq. in.

$\therefore\ AK.KB = (a^2 - b^2)$ sq. in. . . . (i)

Also $R^2 = p^2 + a^2$ (Pythagoras' Th.),

and $x^2 = p^2 + b^2$ (Pythagoras' Th.).

$\therefore$ $R^2 - x^2 = a^2 - b^2$ . . . . . (ii)

From (i) and (ii) it follows that $AK.KB = OA^2 - OK^2$.

Similarly we may prove that $CK.KD = OC^2 - OK^2$.

$\therefore$ $AK.KB = CK.KD = R^2 - OK^2$. *Q.E.D.*

NOTE. If AB and CD meet at K *outside* the circle we may prove that $AK.KB = CK.KD = OK^2 - R^2$ . . . (iii)

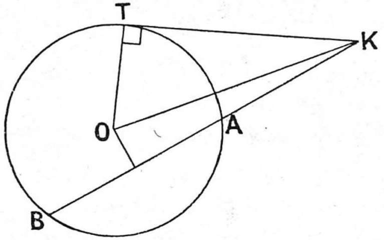


FIG. 377

If from K a tangent KT is drawn to the circle (Fig. 377),

$$OK^2 = OT^2 + KT^2 \quad (\widehat{OTK} = 90°).$$

$$\therefore \quad KT^2 = OK^2 - OT^2.$$

But $KA.KB = OK^2 - OA^2$ (from (iii)).

$$\therefore \quad KT^2 = KA.KB \text{ (since } OT = OA).$$

Thus we have proved the following theorem.

THEOREM 52

If from a point outside a circle a tangent and a secant be drawn, then the square on the tangent is equal to the rectangle contained by the secant and that part of the secant which lies outside the circle.

Further proofs of Theorems 51 and 52 are given on pp. 324–325, and exercises on the theorems on pp. 329–331.

CONSTRUCTION

CONSTRUCTION 19

To divide a straight line into two parts such that the rectangle contained by the whole line and one part shall be equal to the square on the other part.

Analysis. Let AB be the given straight line, say $2a$ in. long.

Let P be the required point such that $AP.AB = PB^2$.

Let $PB = x$ in.

We must have $2a(2a - x) = x^2$,

i.e., $4a^2 = x^2 + 2ax$,

i.e., $5a^2 = (x + a)^2$.

This suggests the following construction.

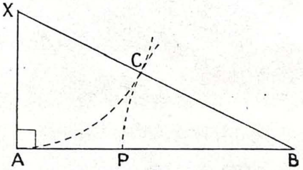


FIG. 378

Given a straight line AB, and let AB be $2a$ in. long.

Construct a point P on AB such that $AP.AB = PB^2$.

Construction. At A draw AX perpendicular to BA making $AX = a$ in.

Draw the circle with centre X and radius XA to cut XB at C.

Draw the circle with centre B and radius BC to cut BA at P.

Then P is the required point.

Proof. We know that

$$BX^2 = AX^2 + AB^2 \quad \text{(Pythagoras' Theorem)}$$
$$= (a^2 + 4a^2) \text{ sq. in.}$$
$$= 5a^2 \text{ sq. in.}$$

Also $BC = x$ in. $(BC = BP)$.

$\therefore\ (BC + XC)^2 = BX^2$ gives

$$(x + a)^2 = 5a^2.$$

$$\therefore\ x^2 = 4a^2 - 2ax.$$

$$\therefore\ x^2 = (2a)(2a - x),$$

i.e., $BP^2 = AB.AP$.

Hence P is a point on AB such that $BP^2 = AB.AP$.

Note. A second point of division, say P_1, can be found on AB produced thus:

Produce BX to cut the circle with centre X and radius XA at C_1.

Draw the circle, centre B, radius BC_1, to cut AB produced at P_1.

Then P_1 is a point such that $AP_1.AB = P_1B^2$.

The proof is left as an exercise for the pupil.

Example. Show how to construct an angle of 36° by a geometric construction.

The steps of the construction are:

(i) Draw any straight line AB.
(ii) Divide AB at X such that $BX.BA = XA^2$.
(iii) Draw a circle, centre A, radius AB.
(iv) Draw a circle, centre B, radius XA. Let the circles cut at C.
(v) Join AC. Then $\hat{A} = 36°$.

The pupil may supply the proof by showing that:

(i) $BC^2 = BX.BA$.
(ii) BC touches the circle through A and C.
(iii) $B\widehat{C}X = \hat{A}$.
(iv) $B\widehat{X}C = A\widehat{C}B$.
(v) $CX = CB$.
(vi) $CX = XA$.
(vii) $A\widehat{C}B = 2\hat{A}$.

RATIO

If two lines are x in. and y in. long respectively they are said to be in the *ratio* $x : y$, or $\frac{x}{y}$.

No ratio can be formed between two quantities which are not of the *same kind*. We cannot form a ratio between 3 tons and 4 pence, or between 3 sq. in. and 4 in. We can, however, find the ratio of 3 sq. ft. to 4 sq. yds. by first expressing these two quantities, which are of the *same kind*, in the same units. The ratio is 3 sq. ft. : 36 sq. ft., and is equal to $\frac{1}{12}$.

PROPORTION

The quantities a, b, c, and d are said to be *in proportion* if $a : b = c : d$, and d is said to be the *fourth proportional* to a, b, and c.

The three quantities a, b, and c are in *continued proportion* if $a : b = b : c$, and c is called the *third proportional* to a and b, and b is called the *mean proportional* to a and c.

NOTE. If in the above

the quantity $a = x$ pounds,

x is the numerical measure of the quantity, and the pounds are the units in which the quantity is measured.

So, if $b = y$ pounds,

$c = z$ ft.,

$d = w$ ft.,

and if $\frac{a}{b} = \frac{c}{d}$,

then $\frac{x}{y} = \frac{z}{w}$ . . . . . . (i)

Since x, y, z, and w are numbers we may deduce from (i), by the ordinary laws of algebra, that

$$\frac{x}{z} = \frac{y}{w} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

$$\text{and} \quad xw = yz \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

The corresponding results applied to a, b, c, and d are, of course, meaningless.

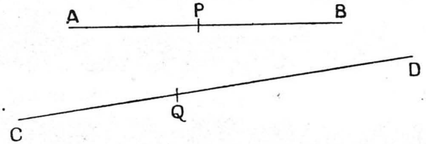


FIG. 379

If a, b, c, and d are the measures of four quantities in proportion, then, by definition,

$$\frac{a}{b} = \frac{c}{d} \quad . \quad . \quad \text{(i)}$$

It may be proved that:

$$\frac{a+b}{b} = \frac{c+d}{d} \quad . \quad \text{(ii)}$$

$$\frac{a+b}{a} = \frac{c+d}{c} \quad . \quad \text{(iii)}$$

$$\frac{b}{a} = \frac{d}{c} \quad . \quad . \quad \text{(iv)}$$

$$\frac{a}{c} = \frac{b}{d} \quad . \quad . \quad \text{(v)}$$

$$\text{and} \quad ad = bc \quad . \quad . \quad \text{(vi)}$$

These results applied to the two straight lines AB and CD shown in Fig. 379 give that if

$$\frac{AP}{PB} = \frac{CQ}{QD} \quad . \quad . \quad \text{(i)}$$

$$\text{then} \quad \frac{AB}{PB} = \frac{CD}{QD} \quad . \quad . \quad \text{(ii)}$$

$$\frac{AB}{AP} = \frac{CD}{CQ} \quad . \quad . \quad \text{(iii)}$$

$$\frac{PB}{AP} = \frac{QD}{CQ} \quad . \quad . \quad \text{(iv)}$$

$$\frac{AP}{CQ} = \frac{PB}{QD} \quad . \quad . \quad \text{(v)}$$

$$\text{and} \quad AP.QD = PB.CQ \quad \text{(vi)}$$

Further it may be proved that if any one of the equalities from (i) to (vi) be given, then the other five are true.

In Fig. 380

P *divides* AB *internally* into the two *segments* AP and PB

whose ratio is $\frac{AP}{PB}$.

P *divides* BA *internally* in the ratio $\frac{BP}{PA}$.

B *divides* AP *externally* in the ratio $\frac{AB}{BP}$.

To show that a given straight line can be divided internally in a given ratio at one point only.

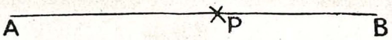


FIG. 380

Suppose P divides the given line AB internally in the ratio $\frac{a}{b}$.

$$\text{Then} \quad \frac{AP}{PB} = \frac{a}{b}.$$

$$\therefore \quad \frac{AP}{AB} = \frac{a}{a+b}.$$

$$\therefore \quad AP = \frac{a}{a+b} \times AB.$$

Since a, b, and AB are given, AP has one value only—*i.e.*, there is one, and only one, position for P.

To show that a given straight line can be divided externally in a given ratio at one point only.

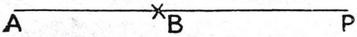


FIG. 381

Suppose P divides the given line AB externally in the ratio $\frac{a}{b}$.

Case (i). If $a > b$, then P lies on AB produced, as in Fig. 381.

$$\text{We have } \frac{AP}{PB} = \frac{a}{b}.$$

$$\therefore \frac{AP}{AP - PB} = \frac{a}{a - b}.$$

$$\therefore AP = \frac{a}{a - b} \times AB.$$

Case (ii). If $a < b$, then P lies on BA produced, and we may show that

$$AP = \frac{a}{b - a} \times AB.$$

In each case there is one, and only one, position for P.

NOTE. See also pp. 356–357, where directed lengths are discussed.

EXERCISES: LIII

1. Find the value of x in each of the following cases:

(i) $4 : 7 = 12 : x$.
(ii) $12 : 17 = x : 68$.
(iii) $12 : x = x : 3$.
(iv) £x : £4 = 4 tons : 6 tons.
(v) 3 yds. : x ft. = 1 sq. ft. : 4 sq. in.

2. Criticize the statement 4 lb. : 7*d*. = 8 lb. : 14*d*.

3. The line AB is 7·2 in. long, and a point P divides it internally in the ratio 3 : 5. Calculate the lengths of the segments AP and PB.

4. Repeat the previous exercise, replacing the word 'internally' by 'externally.'

5. Find the fourth proportional to

(i) 3, 8, 6.
(ii) 4 in., 6 in., 2 sq. ft.

6. Find the third proportional to

(i) 2, 3.
(ii) a, b.

7. Find the mean proportional to

(i) 3, 48.
(ii) a, b.

8. A town A is 60 miles from B and 48 miles from C. On a map the distance from A to B is 24 cm. What is the distance from A to C on the map?

9. The points P and Q divide the line AB, 6·4 cm. long, internally and externally in the ratio 3 : 5. Calculate the length of PQ.

10. A line AB is 10 cm. long. The point P divides AB externally in the ratio 3 : 2, and Q divides BA externally in the ratio 3 : 2. Calculate the length of PQ.

11. A line AB is divided internally at X, Y, and Z so that AX : XY : YZ : ZB = 2 : 3 : 4 : 5. Find $\frac{XZ}{AB}$.

12. The angles of a pentagon ABCDE are such that

$$\widehat{A} : \widehat{B} : \widehat{C} : \widehat{D} : \widehat{E} = 2 : 3 : 4 : 5 : 6.$$

Find the size of each.

13. The points P and Q divide a given line AB of length 4 cm. internally and externally in the ratio 2 : 1. Given that O is the mid-point of AB, find the length of OP and OQ. Verify that $OP.OQ = OA^2$.

14. The points P and Q divide a given line AB internally and externally in the same ratio. Given that O is the mid-point of AB, prove that $OP.OQ = OA^2$.

THEOREM 53

The ratio of the areas of triangles with equal altitudes is equal to the ratio of their bases.

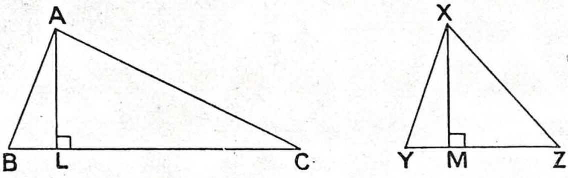


FIG. 382

Given two triangles ABC and XYZ with equal altitudes, AL and XM.

Prove that $\dfrac{\triangle \text{ABC}}{\triangle \text{XYZ}} = \dfrac{\text{BC}}{\text{YZ}}$.

Proof. Let AL $=$ XM $= a$ in.,

BC $= x$ in.,

and YZ $= y$ in.

Then $\triangle$ ABC $= \frac{1}{2}ax$ sq. in.

and $\triangle$ XYZ $= \frac{1}{2}ay$ sq. in.

$$\therefore \frac{\triangle \text{ABC}}{\triangle \text{XYZ}} = \frac{\frac{1}{2}ax}{\frac{1}{2}ay}.$$

$$\therefore \frac{\triangle \text{ABC}}{\triangle \text{XYZ}} = \frac{x}{y};$$

$$i.e., \quad \frac{\triangle \text{ABC}}{\triangle \text{XYZ}} = \frac{\text{BC}}{\text{YZ}}. \qquad Q.E.D.$$

See note on incommensurable lengths on p. 155.

THEOREM 54

If a straight line is drawn parallel to one side of a triangle it cuts the other two sides proportionally.

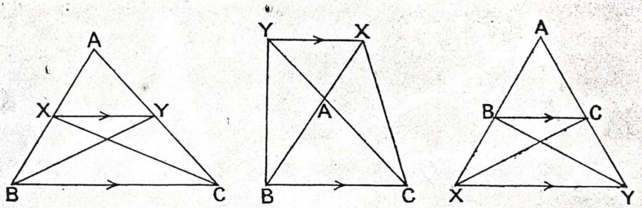


FIG. 383

Given a $\triangle$ ABC with XY parallel to BC.

Prove that $\frac{AX}{XB} = \frac{AY}{YC}$.

Construction. Join YB and XC.

Proof. We know that

$$\frac{AX}{XB} = \frac{\triangle AXY}{\triangle BXY} \quad \text{(common altitude),}$$

and $$\frac{AY}{YC} = \frac{\triangle AXY}{\triangle CXY} \quad \text{(common altitude).}$$

Also $$\triangle BXY = \triangle CXY \quad \text{(base XY, ||s XY, BC).}$$

$$\therefore \frac{\triangle AXY}{\triangle BXY} = \frac{\triangle AXY}{\triangle CXY}.$$

$$\therefore \frac{AX}{XB} = \frac{AY}{YC}.$$

Q.E.D.

THEOREM 55

A straight line which cuts two sides of a triangle proportionally* is parallel to the third side of the triangle.

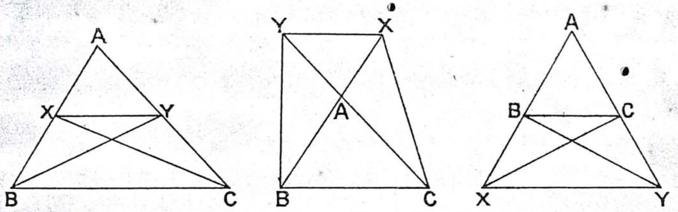


FIG. 384

Given a $\triangle$ ABC with $\frac{AX}{XB} = \frac{AY}{YC}$.

Prove that XY is parallel to BC.

Construction. Join YB and XC.

Proof. We know that

$$\frac{AX}{XB} = \frac{\triangle AXY}{\triangle BXY} \quad \text{(common altitude)},$$

$$\text{and} \quad \frac{AY}{YC} = \frac{\triangle AXY}{\triangle CXY} \quad \text{(common altitude)}.$$

$$\text{Since} \quad \frac{AX}{XB} = \frac{AY}{YC} \quad \text{(given)},$$

$$\therefore \quad \frac{\triangle AXY}{\triangle BXY} = \frac{\triangle AXY}{\triangle CXY}.$$

$$\therefore \quad \triangle BXY = \triangle CXY.$$

These triangles have a common base XY and lie on the same side of XY.

$\therefore$ XY is parallel to BC. *Q.E.D.*

* When we say that XY divides the lines AB and AC proportionally we assume that X and Y both divide AB and AC internally in the same ratio, or both divide AB and AC externally in the same ratio.

EXERCISES: LIV

Section A

1. The area of △ ABC is $7\frac{1}{2}$ sq. in., and P divides BC internally in the ratio 2 : 3. Find the area of △ ABP.

2. The area of △ ABC is 10 sq. in., and P and Q divide BC internally and externally in the ratio 7 : 3. Calculate the area of △ APQ.

3. In the △ ABC the line HK is drawn parallel to BC to meet AB and AC (not produced) at H and K.

(i) Given that AH = 2 in., HB = 3 in., and AK = 2·4 in., find AC.

(ii) Given that $\frac{AH}{AB} = \frac{3}{4}$ and AK = 5·1 in., find KC.

(iii) Given that AC = 7 cm., KC = 4 cm., and AB = 9 cm., find AH.

4. The sides AB and DC of a trapezium ABCD are parallel, and AC and BD meet at E.

(i) $\frac{AE}{AC} = \frac{3}{4}$. Find $\frac{BE}{ED}$.

(ii) BE = 4 cm., BD = 7 cm., and CE = 6 cm. Find AE.

5. The side AB of the parallelogram ABCD is 2 in. long. Through C a straight line is drawn meeting AB produced at E and AD produced at F. If AE = 5 in. and AF = 6 in., what is the length of AD?

6. The area of △ ABC is 15 sq. in., L divides BC internally in the ratio 2 : 3, the line LM parallel to CA meets AB at M, and the line MN parallel to BC meets AC at N. Calculate the area of △ NLC. (*Hint.* Join BN.)

7. In the △ ABC points H and K are taken on AB and AC so that HK is parallel to BC and AH = KC. If AB : AC = $\frac{2}{3}$, what is the ratio $\frac{AH}{HB}$?

8. Points L and M are taken on the sides AB and AC of a △ ABC such that $\frac{AL}{LB} = \frac{CM}{MA} = \frac{1}{3}$, and CX is drawn parallel to ML to meet AB at X. Find $\frac{AX}{XB}$.

9. In the parallelogram ABCD a point L is taken on BC, and AL and DC are produced to meet at M. Prove that $\frac{AL}{AM} = \frac{AB}{DM}$.

10. The parallel straight lines AB, CD, and EF are cut by the transversals ACE and BDF. Prove that $\frac{AC}{CE} = \frac{BD}{DF}$.

Section B

11. Prove that the diagonals of a quadrilateral divide it into four triangles whose areas are in proportion.

12. In the parallelogram ABCD a straight line is drawn through D to meet AC at L, AB at M, CB produced at N, and a parallel to CA through B at P. Prove that $\frac{DM}{MN} = \frac{LP}{PN}$.

13. The straight lines DA, EB, and FC when produced meet at one point. Given that AB is parallel to DE and CB is parallel to FE, prove that AC is parallel to DF.

14. In the △ ABC the mid-point of BC is D. From any point P on BC parallels are drawn to BA and AC cutting AD (produced if necessary) at X and Y. Prove that BXCY is a parallelogram.

15. In the △ ABC, AB = AC, L is any point on BA produced, M is any point on AC produced, and ML cuts BC produced at N. Prove that $\frac{LN}{NM} = \frac{BL}{MC}$. (*Hint.* Draw through M a parallel to CB.)

16. In the △ ABC a point X is taken on BC, and XL and XM are drawn parallel to BA and CA to cut AC and AB at L and M. Prove that △ XLM is a mean proportional to the △s BXM and CXL.

17. A point X is taken on the diagonal BD of a parallelogram ABCD such that $\frac{BX}{XD} = \frac{2}{5}$. Prove that △ ABX = $\frac{1}{7}$ area of parallelogram ABCD.

18. Given in the previous exercise that M is a point of trisection of AD (nearer to A), find what fraction of the area of ABCD is the area of quadrilateral AMXB.

19. A point P is taken inside the △ ABC, and AP produced meets BC at D. Prove that $\frac{\triangle APB}{\triangle APC} = \frac{BD}{DC}$.

20. Use the preceding exercise to prove that the three medians of a triangle meet in a point.

THEOREM 56.

If the interior or exterior vertical angle of a triangle is bisected the bisector divides the opposite side in the ratio of the sides containing the bisected angle.

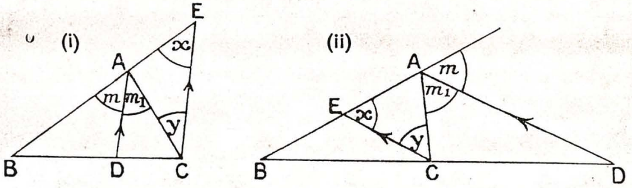


FIG. 385

Given a △ ABC with AD bisecting $\widehat{BAC}$ (*a*) internally as in Fig. 385 (i), (*b*) externally as in Fig. 385 (ii), and cutting BC (produced if necessary) in D.

Prove that $\frac{BD}{DC} = \frac{BA}{AC}$.

Construction. Through C draw CE parallel to DA to cut BA (produced if necessary) in E.

Proof. We know that

$$m = m_1 \quad \text{(given)},$$

$$m = x \quad \text{(corr. angles; AD} \parallel \text{EC)},$$

$$\text{and} \quad m_1 = y \quad \text{(alt. angles; AD} \parallel \text{EC)}.$$

$$\therefore \quad x = y.$$

$$\therefore \quad AE = AC.$$

$$\text{Now} \quad \frac{BD}{DC} = \frac{BA}{AE} \quad (\text{AD} \parallel \text{to side of } \triangle \text{ BEC}).$$

$$\therefore \quad \frac{BD}{DC} = \frac{BA}{AC}.$$

Q.E.D.

THEOREM 57

If a line is drawn through the vertex of a triangle to divide the base, internally or externally, in the ratio of the other two sides of the triangle, then this line bisects the vertical angle of the triangle internally or externally.

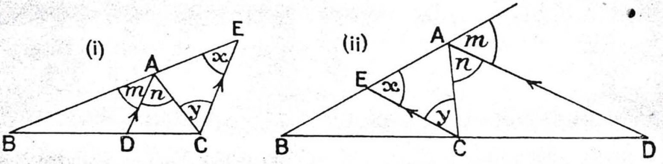


FIG. 386

Given a △ ABC with a line AD drawn through A and meeting BC (*a*) internally at D in Fig. 385 (i), (*b*) externally at D in Fig. 385 (ii), and such that $\frac{BD}{DC} = \frac{BA}{AC}$ in both figures.

Prove that $m = n$.

Construction. Through C draw CE parallel to DA to cut BA (produced if necessary) in E.

Proof. We know that

$$\frac{BD}{DC} = \frac{BA}{AE} \quad \text{(AD is parallel to side of } \triangle \text{ BEC)},$$

$$\text{and} \quad \frac{BD}{DC} = \frac{BA}{AC} \quad \text{(given)}.$$

$$\therefore \quad \frac{BA}{AE} = \frac{BA}{AC}$$

$$\therefore \quad AE = AC.$$

$$\therefore \quad x = y.$$

$$\text{Now} \quad x = m \quad \text{(corr. angles; EC} \parallel \text{AD)},$$

$$\text{and} \quad y = n \quad \text{(alt. angles; EC} \parallel \text{AD)}.$$

$$\therefore \quad m = n.$$

Q.E.D.

Exercises: LV

Section A

1. The internal bisector of $\widehat{ABC}$ of the $\triangle$ ABC meets AC at X.

(i) Given that AB = 4 in., BC = 3 in., and AX = 2 in., find the length of XC.
(ii) Given that AB = 8 cm., BC = 9 cm., and AC = 14 cm., find the length of AX.
(iii) Given that AX = 9·2 in., AC = 13·8 in. and AB = 15 in., find the length of BC.

2. In the $\triangle$ ABC a point X is taken on AC. Does BX bisect $\widehat{ABC}$ when

(i) AB = 4·2 in., BC = 3·9 in., AX = 1·4 in., and XC = 1·3 in.?
(ii) AB = 1·6 in., BC = 1·2 in., AX = 1 in., and AC = 1·6 in.?

3. In the $\triangle$ ABC the bisector of the exterior angle at A meets BC produced at X.

(i) If AB = 3 in., AC = 2 in., and CX = 5 in., what is the length of BX?
(ii) If AB = 7 in., AC = 5 in., and BC = 5·6 in., how long is CX?
(iii) Given that BC = 8 in., CX = 4 in., and AC = 3 in., find the length of AB.
(iv) Why is BC = 8 in., CX = 4 in. and AC = 5 in. not possible?

4. If, in the $\triangle$ ABC, AB = 3 in., BC = 5 in., AC = 6 in., and the internal and external bisectors of $\widehat{B}$ meet AC (produced if necessary) in X and Y, what is the length of XY?

5. In the quadrilateral ABCD the bisector of $\widehat{B}$ meets AC in L, and LM is drawn parallel to CD to meet AD at M. AB = 4 in., BC = 6 in., and AD = 5·6 in. Find the length of MD.

6. If LD of the previous exercise bisects $\widehat{ADC}$, how long is CD?

7. The area of $\triangle$ ABC is 10 sq. cm., AB = 4 cm., AC = 6 cm., and the bisector of $\widehat{A}$ meets BC at D. Find the area of $\triangle$ ABD.

8. The internal and external bisectors of $\widehat{A}$ of the $\triangle$ ABC meet BC (produced if necessary) in XY. How long is CY when BX = 2·4 in. and XC = 1·6 in?

9. The area of a $\triangle$ ABC is 18 sq. cm., and the bisector of $\widehat{B}$

meets the median AD in X. Calculate the area of △ ABX, given that AB = BC.

10. In the △ ABC, AB = 6 in., AC = 8 in., and the median AD = 5 in. The bisector of $\widehat{C}$ meets AD at X. Calculate the length of AX.

11. In the △ ABC the bisector of $\widehat{A}$ meets BC at D, and AB = DC. Prove that AB is the mean proportional to BD and AC.

12. Prove that, if AD is a median of the △ ABC and the bisectors of $\widehat{ADB}$ and $\widehat{ADC}$ meet AB and AC in X and Y, then XY is parallel to BC.

13. A point P is taken on a circle with diameter AB. Q is any point on AB, and R is a point on BA produced such that $\widehat{RPA} = \widehat{APQ}$. Prove that Q and R divide AB internally and externally in the same ratio.

14. Use Theorem 56 to find a point P on the line AB, length 4 in., such that AP : PB = 2 : 3.

15. I is the in-centre of the △ ABC, and AI produced meets BC at D. Prove that $\frac{AI}{ID} = \frac{AB + AC}{BC}$.

16. In the △ ABC the bisector of $\widehat{A}$ meets BC at D, M is the mid-point of BC, and AB > AC. Prove that $\frac{AB + AC}{AB - AC} = \frac{MB}{MD}$.

17. Prove that, if in the quadrilateral ABCD the bisectors of the angles at B and D meet on AC, then the bisectors of the angles at A and C meet on BD.

Section B

18. In the △ ABC, AB = 10 in., BC = 24 in., and the median BD = 13 in. The bisectors of $\widehat{ABD}$ and $\widehat{DBC}$ meet AC in L and M. Calculate the length of LM.

19. A, X, Y, and B are points in order on a straight line, and AX, XY, and YB each subtend an angle of 45° at a point O on the right bisector of AB. Prove that AX is a mean proportional to AB and XY.

20. The diameter of a circle is AB, P is any point on a perpendicular chord CD, and AP and BP are produced to meet the circle in X and Y. Prove that $\frac{CX}{XD} = \frac{CY}{YD}$.

21. The tangent at A to the circle ABC meets BC produced at D. Prove that if BC = CA, then $\frac{AB}{AD} = \frac{AC}{CD}$.

22. Points A, B, C, and D are taken in this order on a circle, AD is produced to E, the bisector of $\widehat{CDE}$ cuts the circle at F, and FB cuts AC at X. Prove that $\frac{AX}{XC} = \frac{AB}{BC}$.

23. In the △ ABC, BC is fixed, $\widehat{A}$ is constant, and D moves on BC so that $\frac{BD}{DC} = \frac{AB}{AC}$. Show that AD always passes through one of two fixed points.

24. Draw the △ ABC in which AB = 6 cm., CA : CB = 3 : 4, and $\widehat{C} = 40°$.

25. Prove that if B and C are fixed points and P moves so that $\frac{BP}{PC}$ = a given ratio, then P lies on the circle on XY as diameter, where X and Y divide BC internally and externally in the given ratio.

26. In the △ ABC the bisector of the interior and exterior angles at A meet BC, produced if necessary, at P and Q. Given that AX is a median of the triangle, prove that $BC^2 = 4XP\,XQ$.

27. Given a circle ABCD on AC as diameter, with B the mid-point of the arc AC and DC : CA = 4 : 3, prove that if BD meets AC at X, then $\frac{AX}{AD} = \frac{5}{7}$.

28. In the △ ABC, $\widehat{B} = 90°$ and D is a point on BC. Given that AB = c in., BD = a in., DC = b in., and that $c^2(b^2 - a^2) = a^2(a + b)^2$, prove that DA bisects $\widehat{BAC}$.

SIMILAR FIGURES

The pupil will have met examples of similar figures.

If a photograph be enlarged the original photograph and the enlargement are similar figures.

Maps of a country drawn to different scales are similar figures.

If a model is made of a ship the original ship and the model are similar solids.

In each case the similar figures are said to have the same 'shape,' and one is an 'enlargement' of the other.

This is expressed in more precise mathematical language in the case of plane polygons as follows:

Two polygons are said to be similar if

(i) the polygons are equiangular *and*

(ii) the ratio of any side of the first polygon to the corresponding side of the second polygon is the same for all pairs of sides.

It is easy to see that equiangular polygons are not necessarily similar—consider a square and a rectangle—and also that polygons in which the ratio of each pair of corresponding sides is the same are not necessarily similar—consider a parallelogram with sides 4 in. and 2 in. and included angle 40°, and a parallelogram with sides 4 in. and 2 in. and included angle 70°.

In the case of the triangle, however, we shall prove that

(i) if the triangles are equiangular, then they are also similar, and

(ii) if the triangles have their sides proportional, then they are similar.

These facts are proved in Theorems 58 and 59. Theorem 60 gives another case of similar triangles.

THEOREM 58

If two triangles are equiangular the triangles are similar.

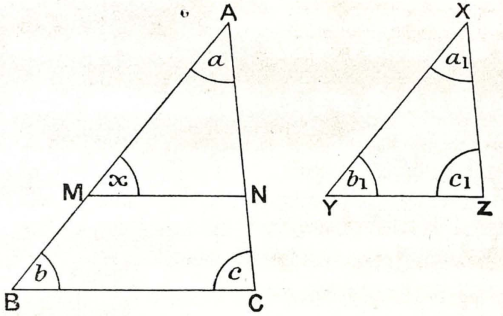


FIG. 387

Given two $\triangle$s ABC and XYZ, with $a = a_1$, $b = b_1$, and $c = c_1$.

Prove that the $\triangle$s ABC and XYZ are similar.

Construction. Along AB cut off AM = XY.
Along AC cut off AN = XZ.

Proof. In the $\triangle$s AMN, XYZ,

$$\begin{cases} \text{AM} = \text{XY} & \text{(construction)}, \\ \text{AN} = \text{XZ} & \text{(construction)}, \\ a = a_1 & \text{(given)}. \end{cases}$$

$\therefore$ $\triangle$ AMN $\equiv$ $\triangle$ XYZ (two sides, incl. angle),

and, in particular,

$$x = b_1.$$

But $b = b_1$ (given).

$\therefore$ $x = b$,

and these are corresponding angles.

$\therefore$ MN $\parallel$ BC.

$$\therefore \frac{AM}{MB} = \frac{AN}{NC} \quad \text{(from } \triangle \text{ ABC).}$$

$$\therefore \frac{AM}{AB} = \frac{AN}{AC}.^*$$

$$\therefore \frac{XY}{AB} = \frac{XZ}{AC} \quad \text{(construction).}$$

Similarly, by cutting off lengths along BA and BC equal to YX and YZ, it can be proved that

$$\frac{XY}{AB} = \frac{YZ}{BC}.$$

$$\therefore \frac{XY}{AB} = \frac{YZ}{BC} = \frac{XZ}{AC}.$$

Also $a = a_1$, $b = b_1$, and $c = c_1$ (given).

$\therefore$ the $\triangle$s ABC and XYZ are similar. *Q.E.D.*

* See p. 297.

THEOREM 59

If the three sides of one triangle are proportional to the three sides of a second triangle the triangles are similar.

First Method

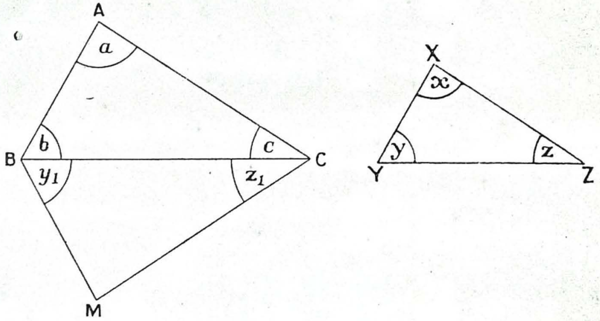


FIG. 388

Given two $\triangle$s ABC and XYZ in which $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$.

Prove that the $\triangle$s ABC and XYZ are similar.

Construction. Make a $\triangle$ BMC with $y_1 = y$ and $z_1 = z$.

Proof. In the $\triangle$s BCM, YZX,

$$y_1 = y \quad \text{(construction)},$$
$$z_1 = z \quad \text{(construction)}.$$
$$\therefore \quad \widehat{M} = \widehat{X} \quad \text{(third angles of triangles)}.$$

Therefore the $\triangle$s are similar (equiangular).

$$\therefore \quad \frac{BC}{YZ} = \frac{BM}{YX}.$$

$$\text{But} \quad \frac{BC}{YZ} = \frac{BA}{YX} \quad \text{(given)}.$$

$$\therefore \quad \frac{BM}{YX} = \frac{BA}{YX}.$$

$$\therefore \quad BM = BA.$$

Similarly CM = CA.

In the $\triangle$s ABC, MBC,

$$\begin{cases} AB = MB & \text{(proved)}, \\ AC = MC & \text{(proved)}, \\ BC = BC. \end{cases}$$

$\therefore$ $\triangle$ ABC $\equiv$ $\triangle$ MBC (three sides),

and, in particular,

$$b = y_1.$$

Since $y_1 = y$ (construction),

$\therefore$ $b = y$.

Similarly $c = z$,

and $a = x$.

$\therefore$ $\triangle$s ABC and XYZ have their sides in proportion (given) and are equiangular (proved).

$\therefore$ $\triangle$s ABC and XYZ are similar. *Q.E.D.*

Second Method

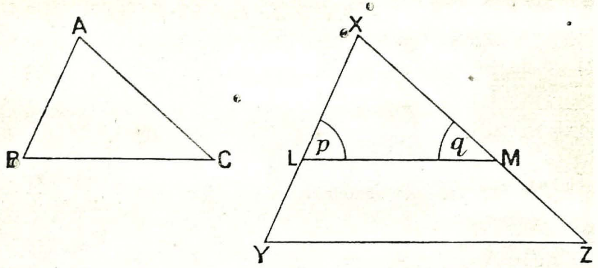


FIG. 389

Given two triangles ABC and XYZ with

$$\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}.$$

Prove that the $\triangle$s ABC and XYZ are similar.

Construction. Along XY make XL = AB.
Along XZ make XM = AC.

Proof. Since $\frac{AB}{XY} = \frac{AC}{XZ}$,

$$\therefore \quad \frac{XL}{XY} = \frac{XM}{XZ}.$$

$\therefore$ LM $\parallel$ YZ (from $\triangle$ XYZ).

In the $\triangle$s XLM, XYZ,

$$\begin{cases} p = \hat{Y} & \text{(corr. angles; LM} \parallel \text{YZ)}, \\ q = \hat{Z} & \text{(corr. angles; LM} \parallel \text{YZ)}, \\ \hat{X} = \hat{X}. \end{cases}$$

Therefore the $\triangle$s are similar (equiangular).

$$\therefore \quad \frac{LM}{YZ} = \frac{XL}{XY}.$$

$$\therefore \quad \frac{LM}{YZ} = \frac{AB}{XY} \quad (XL = AB).$$

$$\therefore \quad \frac{LM}{YZ} = \frac{BC}{YZ} \quad \left(\frac{AB}{XY} = \frac{BC}{YZ}, \text{ given}\right).$$

$$\therefore \quad LM = BC.$$

In the $\triangle$s ABC, XLM,

$$\begin{cases} AB = XL & \text{(construction)}, \\ AC = XM & \text{(construction)}, \\ BC = LM & \text{(proved)}. \end{cases}$$

$$\therefore \quad \triangle ABC \equiv \triangle XLM \quad \text{(three sides)},$$

and, in particular,

$$\hat{B} = p.$$

Since $p = \hat{Y}$,

$$\therefore \quad \hat{B} = \hat{Y}.$$

Similarly $\hat{C} = \hat{Z}$ and $\hat{A} = \hat{X}$.

$\therefore$ $\triangle$s ABC and XYZ have their sides in proportion and are equiangular.

$\therefore$ $\triangle$s ABC and XYZ are similar. *Q.E.D.*

THEOREM 60

If two triangles have an angle of the one equal to an angle of the other and the sides about these equal angles in proportion, then the triangles are similar.

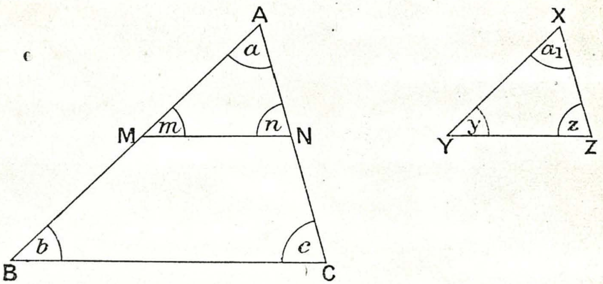


FIG. 390

Given two $\triangle$s ABC and XYZ with $\frac{AB}{XY} = \frac{AC}{XZ}$ and $a = a_1$.

Prove that $\triangle$s ABC and XYZ are similar.

Construction. Along AB and AC cut off AM and AN equal to XY and XZ respectively.

Join MN.

Proof. In the $\triangle$s AMN, XYZ,

$$\begin{cases} AM = XY & \text{(construction)}, \\ AN = XZ & \text{(construction)}, \\ a = a_1 & \text{(given)}, \end{cases}$$

$$\therefore \triangle AMN \equiv \triangle XYZ \quad \text{(two sides, incl. angle)},$$

and, in particular,

$$m = y.$$

$$\text{Now} \quad \frac{AB}{XY} = \frac{AC}{XZ} \quad \text{(given)},$$

$$\therefore \frac{AB}{AM} = \frac{AC}{AN}.$$

$$\therefore \quad MN \parallel BC \quad \text{(MN cuts AB, AC, proportionally)}.$$

$$\therefore \quad m = b \quad \text{(corr. angles)}.$$

Since $m = y$ (proved),

$\therefore \quad b = y.$

Similarly we can prove that

$c = z.$

$\therefore$ $\triangle$s ABC and XYZ are equiangular.

$\therefore$ $\triangle$s ABC and XYZ are similar. *Q.E.D.*

EXERCISES: LVI

Section A

1. In the $\triangle$ ABC, AB = 5 in., BC = 6 in., and CA = 7 in. Along AB a length AD is taken equal to 4 in., and DE parallel to BC cuts AC at E. Calculate the lengths of DE and EC.

2. If in the figure of the preceding exercise DE = 5 in., AE = 6 in., and EC = 3 in., what is the length of BC?

3. Calculate the lengths of the sides of a triangle similar to a triangle whose sides are 7 ft., 9 ft., and 10 ft. in length, given that the shortest side is 4·2 cm.

4. The lines BAC and DAE cut at A. If BA = 2·4 cm., AC = 5·1 cm., DA = 3·4 cm., and AE = 3·6 cm., what is the ratio $\frac{BD}{EC}$?

5. The chords AC and BD of a circle meet at X, and AB = 12 in., CD = 10 in., and AX = 5 in. Find the length of DX.

6. A cone of height 10 in. stands on a circular base, radius 6 in. Find the radius of the circular section cut off from the cone by a plane parallel to the base and 3 in. from the base.

7. Two vertical posts stand 12 ft. and 15 ft. above the ground. The bottom of each post is joined to the top of the other post by tight strings. Find the height of the point of intersection of the strings above the ground.

8. A rectangular table 6 ft. long, 3 ft. 6 in. wide, and 3 ft. high stands with its centre immediately under a light. The short side casts a shadow 4·9 ft. long on the floor. Find the height of the lamp above the table. What is the area of the shadow of the table on the floor?

9. The opposite sides AB and DC of the quadrilateral ABCD are parallel. The diagonals meet at X. Prove that $\triangle$s ABX and CDX are similar.

10. In the pentagon ABCDE, $\widehat{BAC} = \widehat{DAE}$ and $\frac{AB}{AD} = \frac{AC}{AE}$. Prove that $\triangle$s DAB and CAE are similar.

11. Three points A, B, and C are in a straight line, and X and Y are on opposite sides of AC such that $\frac{AX}{AY} = \frac{CX}{BY} = \frac{AC}{AB}$. Prove that $\widehat{AXC} = \widehat{AYB}$.

12. The quadrilateral ABCD is cyclic, and AC meets BD at X. Prove that $\triangle$ ABX is similar to $\triangle$ CDX.

13. A point X is taken on the side AB of a parallelogram ABCD, and DX is produced to meet CB produced at Y. Prove that

$$\frac{DX}{DY} = \frac{AX}{AB}.$$

14. The tangent at C to the circle ABC meets AB produced at T. Prove that $\triangle$s CBT and CAT are similar.

15. What is the locus of a point which moves within a given angle so that the ratio of the perpendiculars from the point to the arms of the angle is constant?

16. In the $\triangle$ ABC, $\widehat{ABC} = 90°$ and BE is an altitude. Prove that

(i) $\triangle$s BAE and BEC are similar.
(ii) $\triangle$s BAE and ABC are similar.

17. A point X is taken on the side BC of a $\triangle$ ABC. Prove that the ratio of the radii of the circum-circles of $\triangle$s AXB and CXA is AB : AC.

18. Perpendiculars BX and CY are drawn on to the tangent at A to the circle BAC, and AD is drawn perpendicular to BC. Show that AD is a mean proportional to BX and CY.

19. Points L and M are taken on the diagonals BD and CA of the parallelogram ABCD such that AL and DM produced meet on BC at P. Prove that if AL = DM, then $\frac{PL}{PM} = \frac{PB}{PC}$.

20. In the $\triangle$ ABC, AB = AC. A point L is taken on AC such that $CB^2 = CL.CA$. Prove that BL = BC.

21. The medians AD and BE of a $\triangle$ ABC meet at G. Prove that

(i) $\triangle$s AGB and EGD are similar.
(ii) $GD = \frac{1}{3}AD$.
(iii) **The three medians of a triangle meet in a point.**

Section B

22. In the trapezium ABCD, AB is parallel to DC, AB = 4 in., BC = 5 in., CD = 9 in., and DA = 3 in. If DA and CB are produced to meet at E, what are the lengths of EC and ED?

23. In the cyclic quadrilateral ABCD the sides AB and DC are produced to meet at P. PB = 7 in., PD = 10 in., and AD = 6 in. Calculate the length of BC.

24. The lines AOB and COD meet at O, $\frac{AO}{OB} = \frac{2}{5}$, and AC is parallel to DB. If the distance of C from AB is 6 cm., what is the distance of D from AB?

25. A point A is taken outside the straight line BCDE in which BC = 6 in. and DE = 2·4 in. A line parallel to BE cuts AB, AC, AD, and AE at X, Y, Z, and W. Given that XY = 3·6 in., calculate the length of ZW.

26. In the quadrilateral ABCD, AB = 3 in., BC = 4·5 in., AC = 4 in., AD = 6 in., and $\widehat{CAD} = \widehat{B}$. Calculate the length of CD.

27. The sides AB and DC of a quadrilateral ABCD are parallel, AB = 4 in., and DC = 3 in. The point E divides BD internally in the ratio 1 : 5, and EF is drawn parallel to BA to meet AC at F. What is the length of EF?

28. In the quadrilateral ABCD, AB = 4 in., AD = DB = 6 in., BC = 1·44 in., DB bisects $\widehat{ABC}$, and the bisector of $\widehat{BAD}$ meets BD at E. Show, by considering the $\triangle$s ABE and EBC, that $\frac{EC}{EA} = \frac{3}{5}$.

29. The centres of two circles of radii 2 in. and 3 in. are A and B. A direct common tangent cuts BA produced at S. Given that AB = 6 in., calculate the length of AS.

30. Show that if in the regular pentagon ABCDE, AD cuts BE at X, then AE is a mean proportional to AD and EX.

31. In the $\triangle$ ABC the bisector of $\widehat{BAC}$ meets BC at D and meets the circle ABC at E. Prove that AB.AC = AE.AD.

32. The altitudes AD and BE of a $\triangle$ ABC meet at H. Prove that BH.HE = AH.HD.

33. Two circles cut at A and B. The tangents at A cut the circles at C and D. Prove that BA is a mean proportional to BC and BD.

34. Prove that the transverse common tangent of two circles divides the line of centres internally in the ratio of the radii of the circles.

35. In the $\triangle$ ACB, $\widehat{ACB} = 90°$, CF is an altitude, and the bisector of $\widehat{ACB}$ cuts AB at P. Prove that $\frac{AP}{PB} = \frac{CF}{FB}$.

36. The points P and Q are fixed, and AB and CD are fixed parallel lines. The point X moves on AB, and QY is drawn parallel to PX to meet CD at Y. Prove that

(i) $\frac{PX}{QY}$ is constant.

(ii) XY passes through a fixed point on PQ.

37. Two circles touch externally at A, and a common tangent touches the circles at X and Y and cuts the line of centres (produced) at S. Prove that $SA^2 = SX.SY$.

38. The sides AB and DC of the quadrilateral ABCD are parallel, and AC cuts BD at L. A line drawn through L parallel to AB meets AD at N and BC at M. Prove that $\frac{1}{LM} = \frac{1}{AB} + \frac{1}{CD}$. Deduce that L bisects MN.

39. The tangent at B to a circle, radius R, cuts two parallel tangents at A and C. Prove that $AB.BC = R^2$.

40. The point P moves on the tangent at the fixed point A on a given circle. AB is a diameter of the circle, and PB cuts the circle again at C. Prove that PB.BC is constant.

41. Show how to find a point P within a given $\triangle$ ABC such that if PX, PY, and PZ are the perpendiculars from P on to the sides of the triangle, then PX : PY : PZ = 1 : 2 : 3.

42. A tangent AD is drawn to a circle of which AB is a diameter. The tangent from D to the circle touches the circle at C and cuts at E the tangent at B. Show that if BD cuts AE at X, then XC is parallel to AD.

43. A point P is taken on the major arc AB of a circle through A and B, and perpendiculars PX, PY, and PZ are drawn to the tangent at A, the tangent at B, and AB respectively. Prove that $PX.PY = PZ^2$.

44. The point O is fixed, P moves on a given circle, and Q divides OP in the ratio 2 : 3. Find the locus of Q.

45. The in-centre of the $\triangle$ ABC is I, and the centre of the circle

which touches BC and also AC and AB, both produced, is I_1. Prove that $AI.AI_1 = AB.AC$.

46. Points X and Y are taken inside and outside the $\triangle$ ABC such that $\widehat{YBC} = \widehat{ABX}$ and $\widehat{BCY} = \widehat{BAX}$. Prove that $\triangle$s ABC and BXY are similar.

47. In the $\triangle$s ABC and $A_1B_1C_1$, $\hat{A} = \hat{A}_1$, $\hat{B} = \hat{B}_1$, and $\hat{C} = \hat{C}_1$.

(i) Prove that $\triangle$s ABO and $A_1B_1O_1$ are similar, where O and O_1 are the circum-centres of the triangles.

(ii) Prove that $\triangle$s ABI and $A_1B_1I_1$ are similar, where I and I_1 are the in-centres.

(iii) Prove that $\triangle$s ABG and $A_1B_1G_1$ are similar, where G and G_1 are the centroids.

(iv) Prove that $\triangle$s ABH and $A_1B_1H_1$ are similar, where H and H_1 are the ortho-centres.

48. A straight line cuts the sides AB, BC, and CA of $\triangle$ ABC, produced if necessary, at Z, X, and Y. Prove that

$$AZ.BX.CY = ZB.XC.YA.$$

49. A variable circle touches two fixed circles, centres A and B, externally at P and Q. Prove that, if PQ and AB are produced to meet at K, then K is a fixed point. (Use the previous exercise.)

50. A square PQML is described in the $\triangle$ ABC having LM along BC, P on AB, and Q on AC. Given that AD is an altitude of the $\triangle$ ABC, prove that $BP(BC + AD) = BC.AD$.

51. In the $\triangle$ ABC, AB = AC and O is the mid-point of BC. A straight line through O meets AB at P and AC produced at R, and CK is drawn parallel to AB to meet PR at K. Prove that

(i) $\triangle POB \equiv \triangle KOC$.

(ii) $\dfrac{BP}{PA} = \dfrac{CR}{RA}$.

(iii) $AB(PA + AR) = 2AP.AR$.

52. In the circle ABCD, B is the mid-point of the arc AC and DB meets AC at X. A point Y is taken on the tangent at A remote from D, such that AY = CX. Prove that $\dfrac{XY}{AC} = \dfrac{AX}{AD}$.

SEGMENTS OF CHORDS OF A CIRCLE

Theorem 51 (Second Proof)

If two chords of a circle, produced if necessary, cut one another, then the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

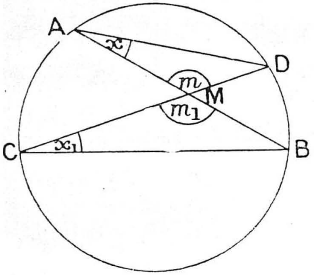


Fig. 391

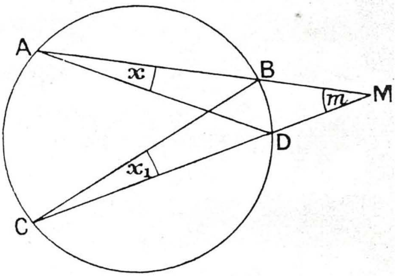


Fig. 392

Given AB and CD two chords of a circle which cut at M.

Prove that AM . MB = CM . MD.

Construction. Join AD and CB.

Proof. In the △s AMD, CMB,

$x = x_1$ (angles in same segment),

$m = m_1$ (vert. opp.) (Fig. 391)

or m is common (Fig. 392).

∴ in each figure,

$\widehat{ADM} = \widehat{CBM}$ (third angles of a △).

∴ △s AMD, CMB, are similar (equiangular).

$$\therefore \frac{AM}{CM} = \frac{MD}{MB}.$$

∴ AM . MB = CM . MD. *Q.E.D.*

THEOREM 52 (SECOND PROOF)

If from any point outside a circle a tangent and a secant be drawn to the circle, then the square on the tangent is equal to the rectangle contained by the whole secant and that part of it which is outside the circle.

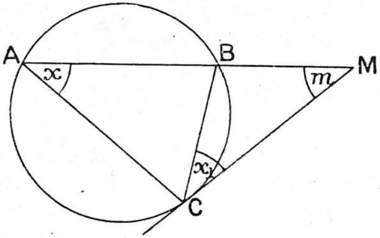


FIG. 393

Given a secant MBA and a tangent MC drawn to the circle CBA.

Prove that MA.MB = MC².

Construction. Join CA and CB.

Proof. In the $\triangle$s MBC, MCA,

$$x_1 = x \quad \text{(tangent, chord theorem)},$$

and m is common.

$$\therefore \widehat{MBC} = \widehat{MCA} \quad \text{(sum of angles of a } \triangle\text{)}.$$

Therefore the triangles are similar (equiangular).

$$\therefore \frac{MB}{MC} = \frac{MC}{MA}.$$

$$\therefore MA.MB = MC^2.$$ *Q.E.D.*

The converses of the above theorems are true. The proofs follow.

Proof of Converse of Theorem 51

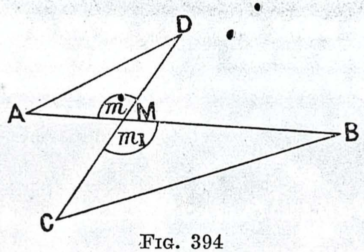


Fig. 394

(I) ***Given*** two intersecting straight lines AMB and CMD such that AM.MB = CM.MD.

Prove that A, B, C, and D lie on a circle.

Proof. In the $\triangle$s AMD, BMC,

$$\frac{AM}{CM} = \frac{MD}{MB} \quad \text{(since AM.MB = CM.MD)},$$

and $m = m_1$.

Therefore the triangles are similar.

$$\therefore \quad \hat{A} = \hat{C},$$

i.e., the straight line BD subtends equal angles at A and C.

$\therefore$ D, B, C, and A lie on a circle. *Q.E.D.*

(II) If BA and DC are produced to meet at M (Fig. 395) and

$$MA.MB = MC.MD,$$

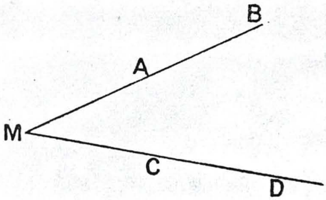


Fig. 395

then A, B, D, and C lie on a circle.

The proof is left to the pupil.

Proof of Converse of Theorem 52

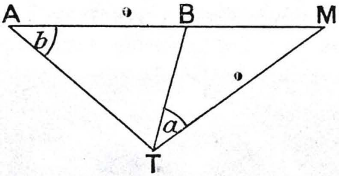


Fig. 396

Given two intersecting straight lines, MT and MBA, such that $MT^2 = MB.MA$.

Prove that MT touches the circle ABT at T.

Proof. In the $\triangle$s AMT, TMB,

$$\frac{MA}{MT} = \frac{MT}{MB} \quad (MT^2 = MB.MA),$$

$$\text{and} \quad \widehat{M} = \widehat{M}.$$

Therefore the triangles are similar.

$$\therefore \quad a = b.$$

Therefore MT touches the circle ABT at T (converse of the tangent, chord theorem). *Q.E.D.*

PTOLEMY'S THEOREM

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.

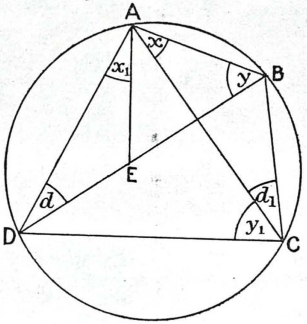


FIG. 397

Given a quadrilateral ABCD inscribed in a circle.

Prove that AC.BD = AB.CD + BC.AD.

Construction. Make $\widehat{DAE} = x$.

Proof. In the $\triangle$s AEB, ADC,

$$y = y_1,$$

$$\text{and} \quad \widehat{EAB} = \widehat{DAC} \quad (\widehat{EAC} + x = \widehat{EAC} + x_1).$$

Therefore the triangles are similar (equiangular).

$$\therefore \quad \frac{AB}{AC} = \frac{BE}{CD}.$$

$$\therefore \quad AB.CD = AC.BE \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

In the $\triangle$s AED, ABC,

$$x_1 = x \quad \text{(construction)},$$

$$\text{and} \quad d = d_1 \quad \text{(angles in same segment)}.$$

Therefore the triangles are similar (equiangular).

$$\therefore \quad \frac{AD}{AC} = \frac{ED}{BC}.$$

$$\therefore \quad AD.BC = AC.ED \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Adding (i) and (ii), we have

$$AB.CD + BC.AD = AC.BE + AC.ED$$
$$= AC\,(BE + ED).$$

$\therefore$ $AB.CD + BC.AD = AC.BD.$ *Q.E.D.*

Exercises: LVII

Section A

1. The chords AB and CD of a circle meet at a point S inside the circle.

(i) If AS = 4 in., SB = 6 in., and SC = 5 in., what is the length of SD?

(ii) If AS = 2 in., AB = 7 in., and CS = 2·5 in., what is the length of CD?

(iii) If AS = 2·4 in., SB = 5 in., and CD = 8 in., how long is CS?

(iv) If AS = 4 in., SB = 6 in., and OS = 5 in. (where O is the centre of the circle), what is the diameter of the circle?

(v) If AS = 2SB, DS = 3 in., and DC = $5\frac{2}{3}$ in., how long is AB?

2. The chords AB and CD of a circle are produced to meet at a point S outside the circle.

(i) Given that SB = 4 in., BA = 2 in., and SD = 4·8 in., find the length of DC.

(ii) Given that SA = 4SB, SD = 3 in., and SC = 12 in., find the length of AB.

(iii) Given that SB = 2·8 in., SA = 5 in., and DC = 5 in., find the length of SD.

3. From a point K outside a circle with centre O a secant KAB is drawn cutting the circle at A and B, and a tangent KT.

(i) If KT = 6 in. and KA = 5 in., how long is KB?

(ii) If KA = 1·6 in. and AB = 2 in., how long is KT?

(iii) If KT = 3·5 in. and AB = 2·4 in., how long is KA?

(iv) If KA = 10 in., AB = 1·9 in., and OA = 5 in., how long is KO?

4. AOB and COD are two intersecting straight lines, and AO = 11 in., OB = 30 in., OC = 15 in., and OD = 22 in. Prove that A, B, C, and D are concyclic points.

5. OAB and OCD are two straight lines, and OA = 4·2 in., OB = 5·5 in., OC = 3 in., and CD = 4·7 in. Prove that A, B, C, and D are concyclic points.

6. OAB and OC are two straight lines, and OC = 2·4 in., OB = 6·4 in., and AB = 5·5 in. Prove that OC is a tangent to the circle ABC.

7. A semicircle is drawn on a line AB as diameter. A point P is taken on AB so that AP = 4 in. From P a perpendicular is drawn to AB cutting the circle at Q. If PQ = 3 in., how long is the radius of the circle?

8. From a point 7 in. from the centre of a circle, radius 5 in., a secant is drawn from which the circle cuts off a chord 5 in. long. Find the lengths of the segments into which the secant is divided.

9. A pendulum is 10 ft. long. As the bob rises vertically 1 ft. from its lowest position through what horizontal distance does it move?

10. A man stands at the top of a mast 50 ft. above sea-level. He sees a ship on the horizon. Taking the radius of the earth to be 4000 miles, find how far he is from the ship. (Answer in miles correct to 1 place of decimals.)

11. A man standing on the top of a cliff 100 ft. high looks out to sea, and over the horizon just sees the top of an island known to be 400 ft. high. Neglecting the height of the man and taking the radius of the earth to be 4000 miles, find how far he is from the top of the island.

12. The altitudes AD, BE, and CF of a triangle ABC meet at H. Prove that AH.AD = AE.AC.

13. Prove that in the figure of the previous exercise

AH.HD = BH.HE = CH.HF.

14. Prove that if two circles intersect their common chord when produced bisects their common tangent.

15. A point P is taken on the common chord (produced) of two circles. Straight lines PAB and PCD are drawn cutting the first circle at A and B, and cutting the second circle at C and D. Prove that ABCD is a cyclic quadrilateral.

16. A circle is drawn to meet two non-intersecting circles. The two common chords are produced to meet at T. Prove that the tangents from T to the three circles are equal.

17. A perpendicular AD is drawn on to the hypotenuse of a right-angled △ ABC. Prove that

(i) AD^2 = BD.DC.
(ii) CA^2 = CD.CB.
(iii) BA^2 = BD.BC.

18. In a fixed circle AB is a fixed diameter. A variable line through A meets the circle at P and meets a fixed line perpendicular to AB at Q. Prove that AP.AQ is constant.

Section B

19. A tower 50 ft. high stands on the top of a cliff and subtends equal angles at two buoys. The distances of the buoys from the foot of the cliff are 250 ft. and 420 ft., and the two buoys and the tower lie in one plane. Find the height of the cliff.

20. The top of a window in a vertical wall is 28·8 ft. and the bottom of the window is 20 ft. from the foot of the wall. How far from the foot of the wall is the point on the horizontal ground at which the window subtends the greatest angle? (*Hint.* Given that A is the top of the window, B the bottom of the window, C the foot of the wall, and D the required point, where A, B, C, and D lie in the same vertical plane, show that CD is a tangent to the circle ABD.)

21. Rugby goal-posts are 18 ft. 6 in. apart. A try is scored just over the goal-line and 30 ft. from the nearest post. Neglecting the height of the cross-bar, find how far from the goal-line the ball should be placed to give the greatest chance of scoring a goal. In what way might the height of the cross-bar affect the answer?

22. A fixed point A is taken on the circumference of a given circle. A variable line through A meets the circle at P, and Q is taken on AP (produced if necessary) so that $AP.AQ =$ constant. What is the locus of Q?

23. A variable circle is drawn through the fixed points A and B, and P is a fixed point on AB produced. Q is the point of contact of a tangent from P to the circle. What is the locus of Q?

24. In the $\triangle$ ABC, $\widehat{ACB} = 2\widehat{BAC}$, AC is produced to D making $CD = CB$. Prove that $DB^2 = DC.DA$.

25. In the cyclic quadrilateral ABCD, $AB = AD$ and the diagonals meet at P. Prove that $AB^2 = AP.AC$.

26. Prove that if APB and CQD are chords of a given circle, and P and Q lie on a concentric circle, then $AP.PB = CQ.QD$.

27. Prove that if three circles intersect the three common chords are either parallel or concurrent.

28. From two points outside a circle tangents are drawn to the circle. Show that the two points and the two mid-points of the chords of contact lie on a circle.

29. ABCD is a semicircle with diameter AD, and AC meets DB at P. Prove that $AD^2 = AP.AC + DP.DB$.

30. The $\triangle$ ABC has $AB = AC$. BA is produced to P making $PC^2 = PA.PB$, and X is any point on PC. Prove that $\widehat{ABC} > \widehat{AXB}$.

AREAS OF SIMILAR TRIANGLES

THEOREM 61

The ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding sides.

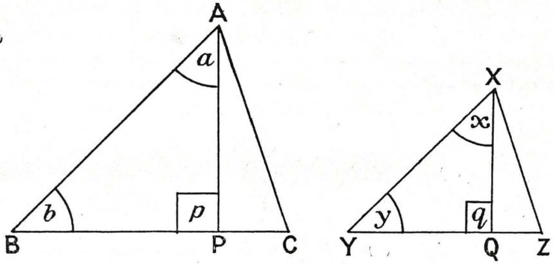


FIG. 398

Given two similar $\triangle$s ABC and XYZ, areas Δ_1 and Δ_2 respectively.

Prove that $\frac{\Delta_1}{\Delta_2} = \frac{BC^2}{YZ^2}$.

Construction. Draw AP and XQ perpendicular to BC and YZ respectively.

Proof. We know that

$$\frac{\Delta_1}{\Delta_2} = \frac{\frac{1}{2}BC.AP}{\frac{1}{2}YZ.XQ}.$$

$$\therefore \quad \frac{\Delta_1}{\Delta_2} = \frac{BC}{YZ} \cdot \frac{AP}{XQ}. \quad \quad (i)$$

In the $\triangle$s ABP, XYQ,

$b = y$ ($\triangle$s ABC, XYZ, similar),

and $p = q$ (both $= 90°$, given).

$\therefore$ $a = x$ (sum of angles of a $\triangle$).

$\therefore$ the $\triangle$s ABP, XYQ, are similar (equiangular).

$$\therefore \quad \frac{AB}{XY} = \frac{AP}{XQ}.$$

We know also that

$$\frac{AB}{XY} = \frac{BC}{YZ} \quad (\triangle s\ ABC, XYZ, \text{similar}).$$

$$\therefore \quad \frac{AP}{XQ} = \frac{BC}{YZ}.$$

$\therefore$ Statement (i) gives

$$\frac{\Delta_1}{\Delta_2} = \frac{BC}{YZ}\cdot\frac{BC}{YZ};$$

$$i.e., \quad \frac{\Delta_1}{\Delta_2} = \frac{BC^2}{YZ^2}. \qquad Q.E.D.$$

Exercises: LVIII

Some of these riders can be done by using the trigonometrical formula $\Delta = \frac{1}{2} bc \sin A$.

Section A

1. Two similar triangles have corresponding sides 2·4 in. and 3 in. long. The area of the first is 3·2 sq. in. Find the area of the second.

2. The areas of two similar triangles are 28 sq. cm. and 63 sq. cm. One side of the first triangle is 8·2 cm. Find the length of the corresponding side of the second triangle.

3. A triangular field has an area of 5 acres. A plan is drawn to scale in which the area of the triangle is 2 sq. ft. What is the scale on which the plan is drawn? How is the result affected if the field is not triangular?

4. The sides AB and CD of the cyclic quadrilateral ABCD are 3 cm. and 4 cm. long. The diagonals meet at L, and the area of $\triangle$ ABL is 3 sq. in. Find the area of $\triangle$ CDL.

5. A point P is taken in the side AB of a $\triangle$ ABC, and PQ is drawn parallel to BC to meet AC at Q. If AP = 2PB, what is the ratio $\dfrac{\triangle APQ}{\triangle ABC}$?

6. The sides BA and CD of the trapezium ABCD are produced to meet at L. The $\triangle$ ALD = $\frac{1}{15}$ trapezium ABCD, and AD = 3·5 cm. Calculate the length of BC.

7. A point L is taken on the side AB of a $\triangle$ ABC. Through L parallels are drawn to BC and AC, cutting AC and BC at M and N. Given that $\dfrac{AL}{LB} = \dfrac{2}{3}$, find the ratio of the areas of the par^{m} LMCN and the $\triangle$ ABC.

8. The sides AB, AC, and BC of a $\triangle$ ABC are 12 in., 5 in. and 13 in. long., and AD is an altitude. Calculate the area of $\triangle$ ABD.

9. In the $\triangle$ ABC, BE and CF are altitudes. If EF = 3 cm. and $\frac{\triangle AFE}{\triangle ABC} = \frac{4}{25}$, what is the length of BC?

10. In the circle ABC, AB is a diameter and AD is drawn perpendicular to the tangent at C. Given that AC = 4 in. and CB = 3 in., find the area of $\triangle$ ACD.

11. Two equilateral triangles are described, one on the side AB of a square ABCD, the other on the diagonal AC. Find the ratio of their areas.

12. In the $\triangle$ ABC a point P is taken on AB such that $\frac{AP}{PB} = \frac{3}{2}$, and PQ is drawn parallel to BC. R is a point on BC such that $\frac{BR}{BC} = \frac{1}{4}$, and AR meets PQ at S. Find $\frac{\triangle APS}{\triangle ABC}$.

13. The sides AB, AC, and BC of the $\triangle$ ABC are 6 in., 7 in., and 8 in. long. The bisectors of the angles BAC and ABC meet the opposite sides at L and M. Through L a line is drawn parallel to BA to meet AC at K. Find $\frac{\triangle CLK}{\triangle CLM}$.

14. In the $\triangle$ ABC, BE and CF are altitudes. Prove that

$$\frac{\triangle ABE}{\triangle ACF} = \frac{AB^2}{AC^2}.$$

15. The chords AB and DC of a circle ABCD are produced to meet at a point E outside the circle. Prove that $\frac{\triangle EAD}{\triangle EBC} = \frac{AD^2}{BC^2}$.

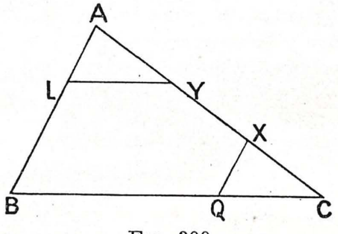


FIG. 399

16. In Fig. 399 the sides of $\triangle$ ABC are trisected as shown. Find the ratio of the areas of the pentagon BLYXQ and the $\triangle$ ABC.

17. In the $\triangle$ ABC, $\widehat{ABC} = 90°$, and BD is an altitude. Prove that $\frac{AB^2}{BC^2} = \frac{AD}{DC}$.

18. In the rectangle ABCD perpendiculars DL and BM are drawn on to AC. Prove that $\frac{AB^2}{AD^2} = \frac{AM}{AL}$.

Section B

19. Prove that the ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding medians.

20. Prove that the ratio of the areas of two similar triangles is equal to the ratio of the squares circumscribing their circumcircles.

21. In the $\triangle$ ABC, $\widehat{ABC} = 90°$. Similar triangles are described having AB, BC, CA, as corresponding sides. Prove that the area of the triangle described on AC is equal to the sum of the areas of the triangles on BC and AB.

22. The altitudes AD, BE, and CF of a $\triangle$ ABC meet at H. Prove that $\frac{\triangle AFH}{\triangle AEH} = \frac{AC^2 . BD}{AB^2 . CD}$.

23. The diameter AB of a circle ABC with centre S is produced to meet at T the tangent at C. The tangent at B meets TC at X. Prove that $\frac{\triangle TBX}{\triangle TCS} = \frac{TB}{TA}$.

24. Prove that the areas of similar polygons are proportional to the squares on corresponding sides.

25. Prove that **if two triangles have an angle of one equal to an angle of the other, then their areas are proportional to the rectangles contained by the sides about the equal angles.**

26. Squares ABXY and ACPQ are described outside the $\triangle$ ABC. Prove that $\triangle$ AXQ = $\triangle$ AYP in area. (*Hint.* Use the result of the previous exercise.)

27. In the quadrilateral ABCD the angles at C and D are right angles. The circle on AB as diameter touches CD at X. Prove that $\triangle$ ABX = $\frac{1}{2}$ quad^{l} ABCD in area.

CONSTRUCTIONS

Construction 20

To construct a fourth proportional to three given lines.

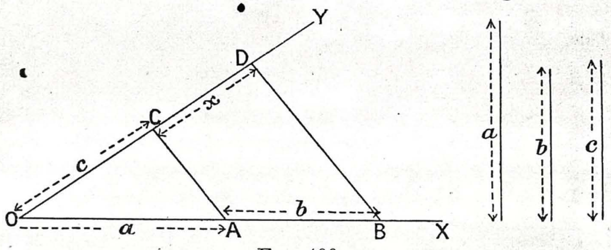


Fig. 400

Given three straight lines a, b, and c.

Construct a line x such that $a : b = c : x$.

Construction. Draw two intersecting lines OX and OY. Along OX make OA $= a$ and AB $= b$. Along OY make OC $= c$. Join AC. Draw BD parallel to AC.

Then CD is the required line.

Proof. We know that in the △ OBD

$$a : b = c : x \quad (\text{AC} \parallel \text{BD}).$$

Note. If in the above construction we replace the length c by the length b, we then get $\frac{a}{b} = \frac{b}{x}$, and so we shall have constructed *a third proportional to the two lines a and b.*

Corollary. *On a given base to construct a rectangle equal in area to a given rectangle.*

Denote the given base by a and the sides of the given rectangle by b and c.

If x is the side of the rectangle we are seeking, then $ax = bc$.

$$\therefore \frac{a}{b} = \frac{c}{x}.$$

Then x is the fourth proportional to a, b, and c, and may be found by the abo[illegible]nstruction.

CONSTRUCTION 21

To construct a mean proportional to two given lines.

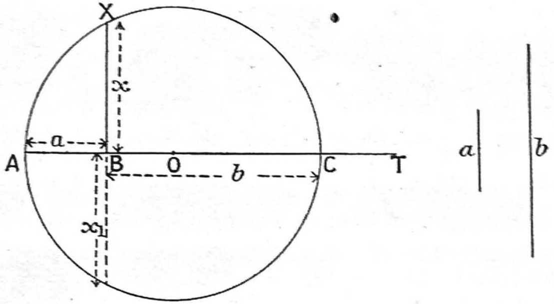


FIG. 401

Given two lines a and b.

Construct a mean proportional to a and b.

Construction. Draw any line AT.

Mark off AB $= a$, and BC $= b$.

Draw the circle on AC as diameter.

Through B draw a line perpendicular to AC meeting the circle at X.

Then BX is the mean proportional to a and b.

Proof. We know that

$$x = x_1 \quad \text{(perp}^{\text{r}}\text{ from centre on to chord).}$$

$$\text{Also} \quad ab = xx_1 \quad \text{(intersecting chords theorem).}$$

$$\therefore \quad ab = x^2.$$

$$\therefore \quad \frac{a}{x} = \frac{x}{b};$$

i.e., x is the mean proportional to a and b.

Corollary I. *To construct a square equal in area to a given rectangle.*

Let the sides of the rectangle be a units and b units long.

Let the side of the square we are seeking be x units long.

Then we want $x^2 = ab$,

$$i.e., \quad \frac{a}{x} = \frac{x}{b}.$$

Hence to get x we construct a mean proportional to a and b.

Corollary II. *To construct a square equal in area to a given polygon.*

By the known construction reduce the polygon, step by step, to a triangle of equal area.

Let the altitude of this triangle be a units long.

Bisect the base of the triangle, getting a line b units long.

Draw a mean proportional to a and b, say x.

Then the side of the square we are seeking is x units long.

Exercises: LIX

1. Find by drawing a fourth proportional to 3 in., 4 in., and 5 in. Check by calculation.

2. Find by drawing a third proportional to 5 and 7.

3. Find by drawing a mean proportional to 6 cm. and 9 cm.

4. Draw a straight line AB, 4 in. long. Divide it internally in the ratio of two lines whose lengths are 5 cm. and 9·3 cm.

5. Draw a straight line AB 1 in. long. Divide it externally in the ratio of two lines whose lengths are 6 cm. and 4·4 cm.

6. Draw a straight line 11 cm. long. Divide it into three parts which are proportional to 4, 3, and 1.

7. Show how to use Theorem 56 to divide a given straight line in a given ratio.

8. Draw a rectangle with sides 4 in. and 3 in. long. Draw (without calculation) a rectangle of equal area one of whose sides is 7 cm. long.

9. Draw a rectangle with sides 4 in. and 3 in. long. Construct a square with the same area.

10. Show how to construct a triangle given its perimeter and the ratio of its sides.

11. Use the mean proportional construction to draw lines whose lengths are (i) $\sqrt{12}$ in., (ii) $\sqrt{35}$ cm.

12. Show how to divide a given arc of a circle so that the chords of the two parts shall be in a given ratio.

13. Draw a quadrilateral ABCD in which AB = 2 in., $\hat{A} = 70°$, AD = 3 in., and BC = DC = 2·5 in. Construct a square having the same area.

14. Draw a regular pentagon with sides 1·5 in. long. Construct a square having the same area.

15. Draw any straight line AB of length a in. Construct a straight line whose length is xa in., where

(i) $x^2 = 5$,
(ii) $(x - 2)^2 = 5$,
(iii) $x^2 - 6x - 11 = 0$.

16. Draw a △ ABC with AB = 8 cm., BC = 7 cm., and CA = 6 cm. Mark a point D on BC such that BD = 3 cm. By a geometric construction draw a circle through C and D to touch the line AB.

17. Show how to use Construction 20 to construct on a given base a triangle whose area shall be equal to that of a given triangle.

Construction 22

To construct a circle which shall pass through two given points and touch a given circle.

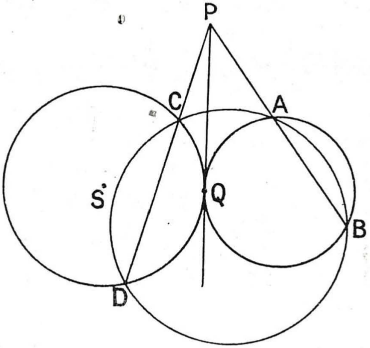


Fig. 402

Given a circle, centre S, and two points A and B.

Construct a circle to pass through A and B and to touch the given circle.

Construction. Draw any circle through A and B to cut the given circle at C and D.

Produce DC and BA to meet at P.

Draw the tangent PQ to the given circle.

Draw the circle through A, B, and Q.

This is the required circle.

Proof. We know that

$PQ^2 = PC.PD$ (tgt., secant theorem, ⊙ CDQ),

and $PC.PD = PA.PB$ (intersecting secant theorem, ⊙ ABC).

$\therefore$ $PQ^2 = PA.PB$.

$\therefore$ PQ is a tangent to circle ABQ.

Hence circles ABQ and CDQ have a common tangent PQ.

Therefore circle ABQ passes through A and B and touches circle CDQ.

Note. There are two solutions, since there are two tangents from P to the given circle.

CONSTRUCTION 23

To cut off from a given triangle, by a line drawn parallel to one side, a triangle whose area shall bear to the area of the first triangle a given ratio.

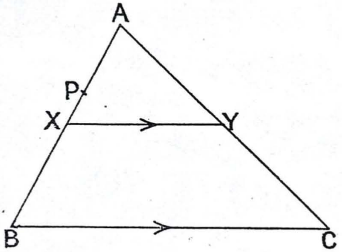


FIG. 403

Let the given ratio be $\frac{3}{7}$.

Given a triangle ABC.

Construct XY parallel to BC such that $\frac{\triangle AXY}{\triangle ABC} = \frac{3}{7}$.

Construction. Divide AB into 7 equal parts.
Let AP contain 3 of these parts.
Construct the mean proportional, say x, to AP and AB.
Make $AX = x$.
Draw XY parallel to BC.
Then XY is the required line.

Proof. Since x is the mean proportional to AP and AB,

$$\therefore \quad x^2 = AP.AB.$$

$$\therefore \quad AX^2 = \tfrac{3}{7}\,AB^2.$$

Also $\triangle$s AXY and ABC are similar (equiangular).

$$\therefore \quad \frac{\triangle AXY}{\triangle ABC} = \frac{AX^2}{AB^2}.$$

$$\therefore \quad \frac{\triangle AXY}{\triangle ABC} = \frac{3}{7}.$$

CONSTRUCTION 24

To construct a polygon similar to a given polygon with an area equal to a given fraction of the area of the given polygon.

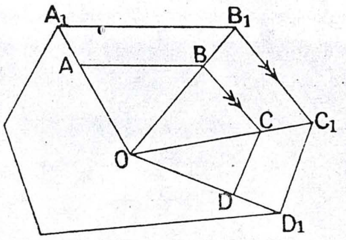


FIG. 404

Let the given fraction be $\frac{5}{7}$.

Given a polygon $A_1B_1C_1D_1 \ldots$

Construct a polygon ABCD ... similar to the polygon $A_1B_1C_1D_1 \ldots$ and such that the ratio of their areas is $\frac{5}{7}$.

Construction. Take any point O inside the polygon $A_1B_1C_1D_1 \ldots$ By Construction 23 draw AB parallel to A_1B_1 so that $\dfrac{\triangle OAB}{\triangle OA_1B_1} = \dfrac{5}{7}$.

Similarly draw BC, CD, etc.

Then polygon ABCD ... is the required figure.

Proof. Since $\widehat{ABO} = \widehat{A_1B_1O}$ (corr. angles; $AB \parallel A_1B_1$),

and $\widehat{CBO} = \widehat{C_1B_1O}$ (corr. angles; $BC \parallel B_1C_1$),

$\therefore \quad \widehat{ABC} = \widehat{A_1B_1C_1}$.

Similarly each angle of polygon ABCD ... is equal to the corresponding angle of polygon $A_1B_1C_1D_1 \ldots$

Also $\dfrac{AB}{A_1B_1} = \dfrac{OB}{OB_1}$ (similar $\triangle$s AOB, A_1OB_1),

and $\dfrac{BC}{B_1C_1} = \dfrac{OB}{OB_1}$ (similar $\triangle$s BOC, B_1OC_1).

$\therefore \quad \dfrac{AB}{A_1B_1} = \dfrac{BC}{B_1C_1}$, etc.

Therefore the polygons ABCD . . . $A_1B_1C_1D_1$ . . . are similar.

Since $\triangle$ OAB $= \frac{5}{7}$ $\triangle$ OA_1B_1 (construction),
and $\triangle$ OBC $= \frac{5}{7}$ $\triangle$ OB_1C_1, etc.,

by addition we have

$$\text{polygon ABCD} \ldots = \tfrac{5}{7} \text{ polygon } A_1B_1C_1D_1 \ldots$$

Example. To construct in a given semicircle a square having one of its sides along a diameter of the semicircle.

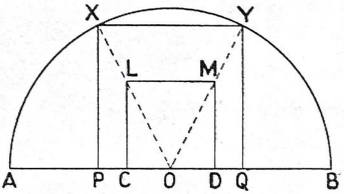


Fig. 405

Given a semicircle with diameter AOB.
Inscribe a square PXYQ having PQ along AB.
Construction. Take any point C on AO.
Make OD = OC.
Complete the square CLMD.
Produce OL and OM to meet the circle at X and Y.
Draw XP and YQ perpendicular to AB.
Then XPQY is the required square.

Proof. We know that

$$\frac{LC}{CO} = \frac{2}{1} \quad (LC = CD),$$

and the $\triangle$s OPX, OCL, are similar (equiangular).

$$\therefore \frac{XP}{LC} = \frac{OP}{OC}.$$

$$\therefore XP = 2OP.$$

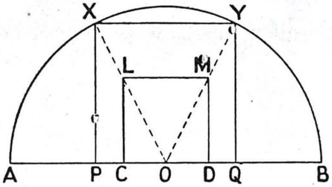


FIG. 405

Also $\triangle LCO \equiv \triangle MDO$ (two sides, incl. angle).

$\therefore \widehat{LOC} = \widehat{MOD}$.

$\therefore \triangle XOP \equiv \triangle YOQ$ (two angles, corr. sides).

$\therefore OP = OQ$.

$\therefore XP = PQ$.

Similarly $YQ = PQ$.

Also $XP \parallel YQ$,

and $XP = YQ$ (each $= PQ$).

$\therefore XY \parallel PQ$.

$\therefore$ XYQP is a parallelogram with one angle $= 90°$.

$\therefore$ XYQP is a square.

EXERCISES: LX (*on Construction* 22)

1. Discuss the solution to Construction 22 when $AB \parallel CD$.

2. Show how to construct a circle to pass through two given points and touch a given straight line.

3. Given a quadrilateral ABCD, find the point in CD (produced if necessary) at which AB subtends the greatest angle.

4. Show how to construct through two given points a circle which cuts off from a given circle a chord of given length.

5. Show how to construct a circle to pass through a given point and to touch two given intersecting straight lines.

6. Show how to construct a circle which passes through a given point A, has its centre on a given straight line BC, and touches a given straight line. (Notice that the image of A in BC must also lie on the circle.)

7. Show how to construct a circle to pass through a given point, have its centre on a given straight line, and touch a given circle.

EXERCISES: LXI

MISCELLANEOUS CONSTRUCTIONS

1. Draw an equilateral triangle with sides 2·5 in. long. Bisect the triangle by a line parallel to one side.

2. Draw a quadrilateral ABCD with AB = 3 cm., $\widehat{B} = 75°$, BC = 5 cm., and AD = CD = 4 cm. Construct a similar quadrilateral LMNP such that $\frac{LM}{AB} = \frac{5}{3}$.

3. Construct a triangle given the base, the ratio of the other two sides, and the vertical angle.

4. Draw a rectangle with sides 3 in. and 2 in. long. Construct a similar rectangle having twice the area.

5. Divide the area of a circle, radius 7 cm., into three equal parts by means of two concentric circles.

6. Construct a rectangle with its sides in a given ratio and having the same area as a given square.

7. Construct a triangle given two of its angles and the length of one median.

8. Construct a triangle given two of its angles and its perimeter.

9. Construct a triangle given two of its angles and the sum of its medians.

10. Construct a parallelogram and trisect it by two lines drawn parallel to a diagonal.

11. Construct in a given quadrant of a circle a square having two of its vertices on the circular arc and one of its vertices on each bounding radius.

12. Construct a triangle given the base, the vertical angle, and the area of the rectangle contained by the other two sides.

13. Show how to inscribe an equilateral triangle in a given triangle.

14. Draw any angle AOB, and take any point P between OA and OB. Find points X and Y on OA and OB such that XPY is a straight line and $\frac{XP}{PY} = \frac{3}{4}$.

15. Show how to construct a triangle given the vertical angle, the perpendicular on to the base, and the ratio of the other two sides of the triangle.

16. Bisect a given triangle by a line drawn perpendicular to the base.

17. Through a given point within a circle construct a chord which is divided in a given ratio at the given point.

18. Justify the following construction for inscribing a square in a given triangle ABC.

(i) Draw the square BCYX.
(ii) Let AX and AY meet BC at L and M.
(iii) Draw the square LMQP.

Then LMQP is the required square (Fig. 406).

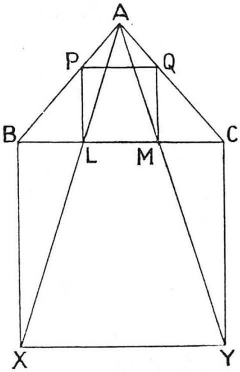


FIG. 406

19. Inscribe a square in a given triangle by the following method: Let the given triangle be ABC. Choose any point X_1 on BC. Draw X_1W_1 perpendicular to BC to cut AB at W_1. Complete the square $X_1W_1Z_1Y_1$ (Y_1 being on BC). Through Z_1 draw a parallel to AC to cut AB at A_1 and BC at C_1. If XWZY is the correct square inscribed in the triangle, where X and Y are on BC, W is on AB, and Z is on AC, what relation holds between BC, BC_1, BX, BX_1? Hence find the position of X.

20. Inscribe a square in a given $\triangle$ ABC by the following method: Using Fig. 406, let BA $= c$ in., AC $= b$ in., BC $= a$ in., the perpendicular from A to BC $= h$ in., and BP $= x$ in. Prove that $x = \dfrac{ac}{a + h}$, and hence obtain BP by drawing a fourth proportional to $a + h$, a, and c.

TEST PAPERS

Paper LXXVII

1. (*a*) If a line AB 5 in. long is divided at C internally in the ratio 3 : 4, what are the lengths of AC and CB?

(*b*) If the same line AB is divided externally in the ratio 2 : 5 at D, what are the lengths of AD and DB?

2. AOB is a straight line, and AO = OB. E is the middle point of OB. Circles are drawn on AE and EB as diameters, and PQL is a common tangent to these circles, meeting them at P and Q and AB produced at L. Prove that BL = the radius of the circle EBQ.

3. ABC is a triangle in which AB = 12 in., BC = 20 in., and CA = 28 in. Find the lengths of the segments into which the sides are divided by the bisectors of the angles of the triangle.

Paper LXXVIII

1. The sides AB, BC, and CA of a triangle are 4 in., 5 in., and 7 in. long. Calculate the length of the projection of CA on AB.

2. Prove that if one diagonal of a trapezium trisects the other diagonal, two of the sides of the trapezium are in the ratio 2 : 1.

3. PQRS is a rectangle, K a point on SP, and the circle on PQ as diameter cuts QK in E. Prove that $\frac{QE}{PQ} = \frac{SR}{QK}$.

Paper LXXIX

1. If in Fig. 407 AX : XB = 8 : 7, and XY represents 12 in. Find the length represented by BC and the ratio $\frac{\triangle AXY}{\triangle ABC}$.

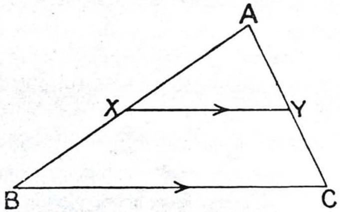


Fig. 407

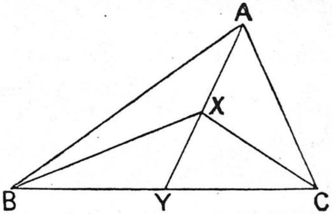


Fig. 408

2. Prove that in Fig. 408 $\frac{BY}{YC} = \frac{\triangle ABX}{\triangle AXC}$.

3. The bisectors of two opposite angles of a quadrilateral meet on a diagonal. Prove that the sides of the quadrilateral are in proportion.

Paper LXXX

1. The sides of a parallelogram are 6 cm. and 7 cm. long. Given that one diagonal is 8 cm. long, find the length of the other diagonal to the nearest millimetre.

2. BC is a median of a triangle ABD, and XY is parallel to AD, X being on AB and Y on DB. Given that XK and YM are both parallel to BC (K and M being on AD), prove that AK = MD.

3. Find a point P within a △ ABC such that the areas of △s APB, BPC, and CPA are all equal.

Paper LXXXI

1. ABC and DBF are two straight lines which cut at B. AB = 4 in., BC = 5 in., DB = 8 in., and BF = $2\frac{1}{2}$ in. Prove that

(i) △s ABD and CBF are similar.
(ii) A, F, C, and D are concyclic points.

2. BC is a chord of a circle, and BC = 11 in. D is a point on BC 8 in. from B. E is the middle point of the arc BC, and ED produced cuts the circle again in A. Find the ratio AB : AC.

3. A triangle XYZ is inscribed in a circle. A line YM is drawn parallel to the tangent at X to meet XZ in M. Prove that

$$XY.YZ = XZ.YM.$$

Paper LXXXII

1. AD and AE are the internal and external bisectors of $\widehat{A}$ of △ ABC, and they meet BC and BC produced in D and E respectively. If AB = 6 in., AC = 8 in., and BC = 10 in., what is the length of DE?

2. ABC and DEF are two isosceles triangles. AB = AC = 6 in., DE = DF = 10 in., and $\widehat{ABC} = \widehat{DEF}$. Given that BC = 4 in., find the length of EF and the ratio $\frac{\triangle ABC}{\triangle DEF}$.

3. Construct a mean proportional between 9 cm. and 6 cm. Measure the constructed proportional. Check your result by calculation.

PAPER LXXXIII

1. Two circles touch internally at A. A line APQ is drawn to cut the circles in P and Q. Prove that $\frac{AP}{AQ} = \frac{r_1}{r_2}$, where r_1 and r_2 are the radii of the two circles.

2. AOB and POQ are two intersecting straight lines, and AO = 2OB and PO = 2OQ. Prove that AP = 2BQ, and find the ratio △ AOP : △ BOQ.

3. Find a point Q on the side XZ of a △ XYZ such that

$$\frac{XQ}{XY} = \frac{XY}{XZ}.$$

PAPER LXXXIV

1. In Fig. 409 PT is a tangent.

(i) Given that AB = 2 cm. and BP = 6 cm., find PT.
(ii) Given that PT = 8 cm. and PB = 3 cm., find PA.
(iii) Given that TB = 5 cm., TA = 7 cm., and PT = 6 cm., find PA.

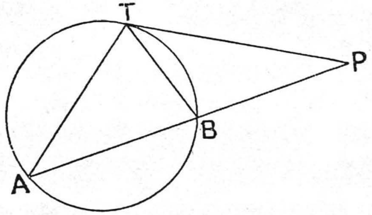


FIG. 409

2. A man 6 ft. 2 in. high casts a shadow 4 ft. long. At the same moment a boy accompanying him casts a shadow 3 ft. 3 in. long. How tall is the boy?

3. ABC is a triangle, and O is a point outside it. D and E are points on OA and OB such that $\frac{OD}{DA} = \frac{OE}{OB} = \frac{1}{2}$. DF is drawn parallel to AC to meet OC in F. Prove that △s ABC and DEF are similar and find $\frac{\triangle DEF}{\triangle ABC}$.

PAPER LXXXV

1. ABCD is a trapezium, AB being parallel to DC, and its diagonals meet at E. If AB = 3 in., DC = 5 in., what is the ratio

of the area of △ AEB to the area of △ DEC? If in addition BE = $1\frac{1}{2}$ in. and CE = 4 in., how long are AE and DE?

2. BE and CF are altitudes of the △ ABC. Prove that △s AFE and ABC are similar.

3. XYZ is a triangle right-angled at Z. DEFG is a square with D and E on XY, G on XZ, and F on YZ. Prove that the area of this square equals rectangle XD.EY.

Paper LXXXVI

1. Draw a line AB 4 in. long. By a geometrical construction find a point X in AB such that AX = 6XB.

2. In a triangle ABX a line DE is drawn such that $\frac{AD}{DB} = \frac{AE}{EX}$, D being on AB and E on AX. Prove that if BX is produced its own length to C and if DE produced cuts AC in F, then $\frac{AB}{AD} = \frac{XC}{EF}$.

3. BC is the diameter of a circle and AD a parallel chord. The tangent at C cuts AD produced at E. Prove that $\frac{BC}{BD} = \frac{BD}{AE}$.

Paper LXXXVII

1. A point P is taken outside a given circle, centre O. Tangents PQ and PR are drawn to the circle, and QR meets OP at K. Prove that $\frac{OK}{OQ} = \frac{OQ}{OP} = \frac{QK}{PQ}$.

2. Given that the bisectors of $\hat{A}$ and $\hat{C}$ of a quadrilateral ABCD meet on BD, prove that the bisectors of $\hat{B}$ and $\hat{D}$ meet on AC.

3. XY is a diameter of a circle, and tangents are drawn at X and Y. The tangent at another point A on the circle cuts the first two tangents at B and C respectively. BY and CX meet at F. Prove that AF is parallel to BX.

Paper LXXXVIII

1. ACB is an arc of a circle. Find by a geometrical construction the position of C when the chord AC = 2CB.

2. Two equal circles cut at X and Y. The tangent at X to one circle cuts the other at P, and PY (produced if necessary) cuts the first circle at Q. Prove that △s XPY and PXQ are similar isosceles triangles.

3. In the convex quadrilateral ABCD, AB = 2 in., BC = 3 in., CD = 4 in., and DA = 5 in. Perpendiculars BX and DY are drawn on to AC. Prove that AC.XY = 7 sq. in.

Paper LXXXIX

1. Prove that the ratio of the areas of similar quadrilaterals is the ratio of the squares on corresponding diagonals.

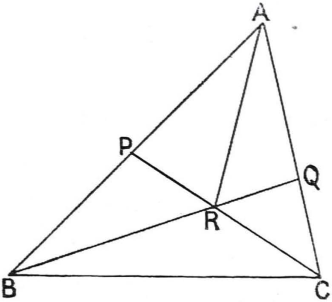
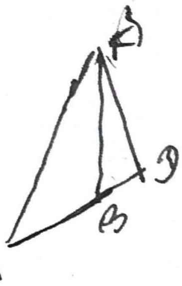


Fig. 410



2. In Fig. 410 AP = PB and AQ = 2QC. Find

(i) the ratio of the areas of $\triangle$s ARC and CRQ.

(ii) $\frac{BR}{RQ}$ and $\frac{PR}{RC}$.

3. In a $\triangle$ ABC, $\hat{B}$ is obtuse and AD is an altitude. Prove that if $AC^2 - AB^2 = 7BC^2$, then DB = 3BC.

Paper XC

1. Light from an electric bulb shines through a hole 3 in. in diameter in a wall and falls on a parallel wall 6 ft. away. If the distance from the light to the first wall is 5 ft., what is the diameter of the lighted part of the second wall?

2. Prove that if two medians of a triangle are equal the triangle is isosceles.

3. Show how to construct the $\triangle$ ABC when AB, $\widehat{C}$, and $\frac{AC}{CB}$ are given.

Paper XCI

1. The internal and external bisectors AP and AQ of $\widehat{A}$ of $\triangle$ ABC meet BC, produced if necessary, in P and Q respectively. M is the foot of the perpendicular from A to BC. Prove that PM : AM = AM : MQ.

2. Draw, by a geometrical construction, a line $\sqrt{15}$ in. long.

3. X and Y are two fixed points, and PQ and CD are two fixed parallel straight lines. X is joined to any point M on PQ, and YN is drawn parallel to XM to meet CD in N. By showing that $\frac{YN}{XM}$ = constant, prove that MN always passes through a fixed point.

Paper XCII

1. APB is a straight line, and A, P, and B are fixed points on it. XY is a given straight line. Find a point Q on XY, produced if necessary, such that PQ bisects $\widehat{AQB}$.

2. In the $\triangle$ ABC, P is on BC and $\frac{BP}{PC} = \frac{1}{2}$, Q is on AC and $\frac{CQ}{QA} = \frac{5}{4}$, and R is on AB and $\frac{AR}{RB} = \frac{2}{1}$. Find $\frac{\triangle PQR}{\triangle ABC}$.

3. Given that R is the radius of circle ABC, and AD is an altitude of the $\triangle$ ABC, prove that $R = \frac{AB.AC}{2AD}$.

Paper XCIII

1. A point P moves on the circumference of a fixed circle. Q is a fixed point outside the circle, and QP is divided at R in a fixed ratio. Find the locus of R.

2. On a given circle are two fixed points A and B. P is a variable point on the circle. Points E and D are chosen on PA and PB respectively so that $\frac{PE}{PB} = \frac{PD}{PA}$ = constant. Prove that ED is fixed in length.

3. In the $\triangle$ ABC, $\widehat{B}$ is acute and AD is an altitude. Given that $AC^2 - AB^2 = \frac{1}{3}BC^2$, prove that DB = $\frac{1}{3}$BC.

PAPER XCIV

1. R is the mid-point of the side AB of $\triangle$ ABC. A line RPQ is drawn to cut BC at P and AC produced at Q. Prove that

$$\frac{BP}{PC} = \frac{AQ}{QC}.$$

2. ABCD in Fig. 411 is a parallelogram. Prove that

$$RP.QP = AP^2.$$

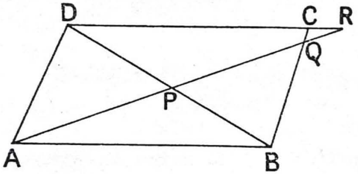


FIG. 411

3. In the $\triangle$ ABC, BE is a median. Given that AB : BE : BC = 2 : 2 : 3, prove that $2AC^2 = 5AB^2$.

PAPER XCV

1. In a right-angled $\triangle$ ABC, BD is a perpendicular on to the hypotenuse AC and BD is produced to E so that BD : DC = DC : DE. Prove that $\triangle$s AED and BCD are equal in area.

2. Three equal circles, radius r, touch each other. Their centres are A, B, and C. Prove that if BC is produced to cut one of the circles at D, then $AD^2 = 7r^2$.

3. ABCD is a quadrilateral with diagonals AC and BD. Prove that the quadrilateral formed by joining the centroids of $\triangle$s ABD, BCD, ADC, and ABC is a similar quadrilateral of one-ninth the area of ABCD.

PAPER XCVI

1. DE is a line parallel to the base BC of a triangle ABC, cutting AB and AC in D and E. Prove that if F is the point of intersection of DC and EB, then AF produced bisects BC.

2. The points P and Q are fixed on a given circle, and M is any

point on the minor arc PQ. The bisector of $\widehat{PMQ}$ meets the circle at X. Prove that $\frac{PM + QM}{MX}$ = constant. (*Hint.* Use Ptolemy's Theorem, pp. 328–329.)

3. AB is a straight line and P a point on one side of it which moves so that AP : PB is constant. What is the locus of P?

Paper XCVII

1. Construct a triangle whose sides are in the ratio 3 : 5 : 6 in a circle of radius 2 in.

2. Prove that if three circles intersect one another, then their three common chords are either concurrent or parallel.

3. A point P is taken inside the rectangle ABCD such that AB and CD subtend supplementary angles at P. Lines AR, BR, are drawn parallel to DP and CP respectively. Prove that

(i) ARBP is a cyclic quadrilateral.
(ii) PA.PC + PB.PD = area of rect. ABCD.

(*Hint.* Use Ptolemy's Theorem, pp. 328–329.)

Paper XCVIII

1. The △ ABC is right-angled at A. Squares AKJB and AFGC are described on the sides AB and AC. Show that KF, AG, and BC meet in a point.

2. In the △s ABC and XYZ, AB = XY, AC = XZ, and the median through A is equal in length to the median through X. Prove that the triangles are congruent.

3. Find, by a geometric construction, a point whose distances from three fixed points are p in., q in., and r in., where

$$p : q : r = 1 : 2 : 3.$$

(*Hint.* Use Exercise 3 of Paper XCVI.)

Paper XCIX

1. On the sides AB, BC, and CA of a right-angled triangle similar polygons are described, the sides of the triangle being corresponding sides of the polygons. Show that the sum of the areas of two of the polygons is equal to the area of the third polygon.

2. In the △ ABC, AB = AC and O is the mid-point of BC. Any

line through O cuts AB at L and AC produced at M. Prove that $AB = \frac{2AL.AM}{LA + AM}$. (*Hint.* Draw a parallel through C to AB.)

3. In the $\triangle$ ABC the bisector of $\widehat{BAC}$ meets BC at X. Given that AB : AC = 2 : 3, prove that

$$3AB^2 + 2AC^2 = 3BX^2 + 2XC^2 + 5AX^2.$$

Paper C

1. In the $\triangle$ ABC the bisector of $\widehat{BAC}$ meets BC at D. Prove that $AB.AC = AD^2 + BD.DC$. (*Hint.* Produce AD to cut the circle ABC.)

2. The diagonals of a quadrilateral ABCD meet at O. The centroids of the $\triangle$s AOB, BOC, COD, and DOA are P, Q, R, and S. Prove that PQRS is a parallelogram and that its area is two-ninths of the area of the quadrilateral ABCD.

3. Use the construction on p. 294 to construct an isosceles triangle having each of the equal angles double the third angle.

PART VI

MISCELLANEOUS THEOREMS

DIRECTED LENGTHS

So far we have attached no sign to the length of the segment AB. Consider the straight line ABC (Fig. 412), where

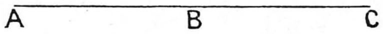


Fig. 412

AB $= a$ in. and BC $= b$ in. If we agree to attach a positive sign to lengths measured in the direction A $\rightarrow$ C, then we define lengths measured in the direction C $\rightarrow$ A to be negative.

With this definition $\quad \text{AB} = + a$ in.,

$\text{BA} = - a$ in.,

and $\quad \text{AB} + \text{BA} = 0.$

Similarly $\text{AB} + \text{BC} + \text{CA} = 0.$

With this sign convention many theorems may be enunciated in a more general form.

Thus, Theorems 48 and 49 may be included in the following single enunciation:

If in any $\triangle$ ABC (see Fig. 413) AD is perpendicular to BC, then, having regard to the sign of the segments,

$$\text{AC}^2 = \text{AB}^2 + \text{BC}^2 - 2\text{BC}.\text{BD}.$$

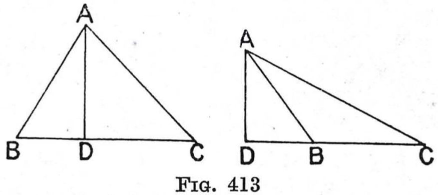


Fig. 413

For, whatever sign we attach to BC,

BD will have the same sign when $\widehat{ABC}$ is acute,

and BD will have the opposite sign when $\widehat{ABC}$ is obtuse.

Euler's Theorem

If A, B, C, and D are four points in order on a straight line, then

$$\mathbf{AB.CD + AC.DB + AD.BC = 0.}$$

Proof.

Let AB $= b$ in., AC $= c$ in., and AD $= d$ in.

Then CD $= (d - c)$ in.,

DB $= (b - d)$ in.,

and BC $= (c - b)$ in.

$\therefore$ AB.CD + AC.DB + AD.BC

$= b(d - c) + c(b - d) + d(c - b)$ sq. in.,

$= (bd - bc + cb - cd + dc - db)$ sq. in.,

$= 0.$ *Q.E.D.*

Menelaus' Theorem

If a straight line cuts the sides BC, CA, and AB of a triangle in X, Y, and Z, then

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

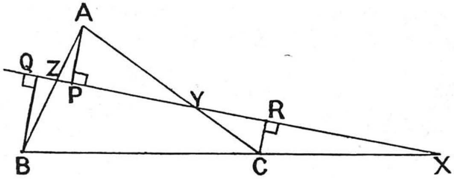


Fig. 414

Given a $\triangle$ ABC and a straight line XYZ cutting the sides in X, Y, and Z.

Prove that $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$

Construction. Draw AP, BQ, and CR each perpendicular to XYZ.

Proof. The transversal XYZ must cut either one side or three sides of the $\triangle$ ABC externally.

Therefore, either one of the ratios $\frac{BX}{XC}$, $\frac{CY}{YA}$, and $\frac{AZ}{ZB}$ is negative or all three ratios are negative.

Therefore their product is negative in each case.

The $\triangle$s BQX, CRX, are similar (equiangular).

$$\therefore \quad \frac{BX}{XC} = \frac{BQ}{CR}.$$

Similarly $$\frac{CY}{YA} = \frac{CR}{AP},$$

and $$\frac{AZ}{ZB} = \frac{AP}{BQ}.$$

$$\text{Since} \quad \frac{BQ}{CR} \cdot \frac{CR}{AP} \cdot \frac{AP}{BQ} = 1,$$

$$\therefore \quad \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1. \qquad Q.E.D.$$

EXERCISES: LXII

1. In the △ ABC, $\frac{AC}{CB} = \frac{2}{3}$, and the bisector of $\widehat{ACB}$ meets AB in Z. If Y is the mid-point of AC and ZY is produced to meet BC produced at X, what is $\frac{CX}{BX}$?

2. Prove the converse of Menelaus' Theorem.

3. Use Menelaus' Theorem to show how to construct a ratio $= \frac{ab}{xy}$.

4. The external bisector of $\widehat{BAC}$ in the △ ABC meets BC at X, and the bisectors of the interior angles at B and C meet the opposite sides of the triangle at Y and Z. Prove that X, Y, and Z are collinear.

5. The tangents at A, B, and C to the circle ABC meet BC, CA, and AB at X, Y, and Z. Prove that X, Y, and Z are collinear.

6. A transversal cuts the sides AB, BC, and CA of a triangle externally at Z, X, and Y. Prove that the product of the tangents from X, Y, and Z to the circle ABC is numerically equal to CY.BX.AZ.

7. A circle ABCDEF is drawn to cut the sides XY, YZ, and ZX of a △ XYZ in A, B; C, D; and E, F, respectively. FA and ZY are produced to meet at L; BC and XZ are produced to meet at M; ED and XY are produced to meet at N. Prove that

$$\text{(i)} \quad \frac{YL}{LZ} \cdot \frac{ZF}{FX} \cdot \frac{XA}{AY} = -1.$$

$$\text{(ii)} \quad \frac{ZM}{MX} \cdot \frac{XB}{BY} \cdot \frac{YC}{CZ} = -1.$$

$$\text{(iii)} \quad \frac{XN}{NY} \cdot \frac{YD}{DZ} \cdot \frac{ZE}{EX} = -1.$$

Deduce that $\frac{YL}{LZ} \cdot \frac{ZM}{MX} \cdot \frac{XN}{NY} = 1$, and hence prove Pascal's Theorem—namely: *The three points of intersection formed by producing the opposite sides of a cyclic hexagon are collinear.*

Ceva's Theorem

If the lines joining any point to the vertices A, B, and C of a triangle meet the opposite sides at X, Y, and Z, then

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=+1.$$

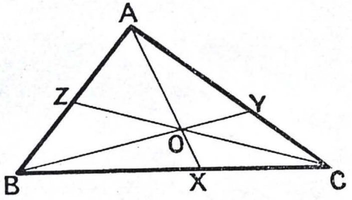


Fig. 415

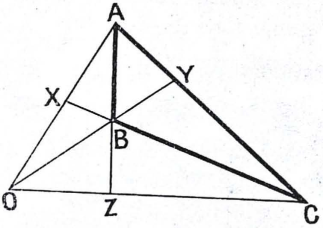


Fig. 416

Given a $\triangle$ ABC and a point O. AO, BO, and CO meet the sides at X, Y, and Z.

Prove that $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=+1.$

Proof. If the lines AX, BY, and CZ meet inside the triangle each of the ratios $\frac{BX}{XC}$, $\frac{CY}{YA}$, and $\frac{AZ}{ZB}$ is positive (see Fig. 415).

If the lines meet outside the triangle two of the ratios are negative and one is positive. (In Fig. 416 $\frac{AZ}{ZB}$ and $\frac{BX}{XC}$ are each negative, and $\frac{CY}{YA}$ is positive.)

$\therefore\ \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}$ is positive in all cases.

Now $\frac{AZ}{ZB}=\frac{\triangle ACZ}{\triangle BCZ}$ (the $\triangle$s have a common altitude),

and $\frac{AZ}{ZB}=\frac{\triangle AOZ}{\triangle BOZ}$ (the $\triangle$s have a common altitude).

$$\therefore \quad \frac{AZ}{ZB} = \frac{\triangle ACZ - \triangle AOZ}{\triangle BCZ - \triangle BOZ} \quad \text{in Fig. 415,}$$

$$\text{and} \quad \frac{AZ}{ZB} = \frac{\triangle ACZ + \triangle AOZ}{\triangle BCZ + \triangle BOZ} \quad \text{in Fig. 416.}$$

$$\therefore \quad \frac{AZ}{ZB} = \frac{\triangle AOC}{\triangle BOC} \quad \text{in both figures.}$$

$$\text{Similarly} \quad \frac{BX}{XC} = \frac{\triangle AOB}{\triangle AOC},$$

$$\text{and} \quad \frac{CY}{YA} = \frac{\triangle BOC}{\triangle AOB}.$$

$$\text{Since} \quad \frac{\triangle AOB}{\triangle AOC} \cdot \frac{\triangle BOC}{\triangle AOB} \cdot \frac{\triangle AOC}{\triangle BOC} = 1,$$

$$\therefore \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Q.E.D.

Exercises: LXIII

1. Use Ceva's Theorem to prove that:

(i) The medians of a triangle concur.
(ii) The bisectors of the angles of a triangle concur.
(iii) The altitudes of a triangle concur.

2. Points P and Q are taken on the sides AB and AC of a triangle such that PQ is parallel to BC. PC and BQ meet at X. Prove that AX produced bisects BC.

3. Prove the converse of Ceva's Theorem.

4. A circle cuts the sides AB, BC, and CA of a triangle at Z and Z_1, X and X_1, and Y and Y_1. Prove that if CZ, AX, and BY concur, then CZ_1, AX_1, and BY_1 concur.

5. The altitudes of a △ ABC are AD, BE, and CF. Prove that the perpendiculars from A, B, and C on to EF, FD, and DE concur.

6. The escribed circles of a △ ABC meet the sides BC, CA, and AB at X, Y, and Z respectively. Prove that

(i) $2BX = AC + BC - BA$.
(ii) AX, BY, CZ, meet in a point.

THE SIMSON LINE

Given a point P on the circum-circle of a △ ABC and PX, PY, and PZ perpendicular to BC, CA, and AB respectively.

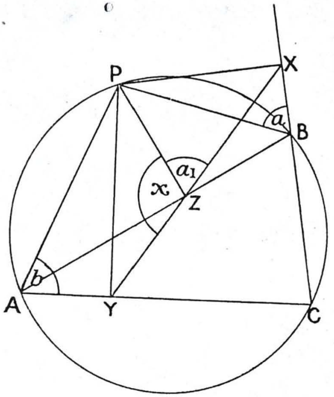


FIG. 417

Prove that XYZ is a straight line (called the Simson Line of the △ ABC with respect to the point P).

Proof. We know that PXBZ is cyclic ($\widehat{PXB} + \widehat{PZB} = 180°$).

$\therefore \quad a = a_1$ (subtended by PX).

Also $\quad a = b$ (ext. angle of cyclic quad[l] PBCA).

PZYA is also cyclic ($\widehat{PZA} = \widehat{PYA}$).

$\therefore \quad b + x = 180°$ (opp. angles of cyclic quad).

$\therefore \quad a + x = 180°$.

$\therefore \quad a_1 + x = 180°$.

Hence XZY is a straight line. *Q.E.D.*

EXERCISES: LXIV

1. Prove the converse of the Simson Line theorem—namely: If P is a point outside a △ ABC such that the feet of the perpendiculars from P on to AB, BC, and CA are collinear, then PABC is a cyclic quadrilateral.

2. If P is any point on the circum-circle of a △ ABC in which $\widehat{B} = 90°$, prove that PB is bisected by the Simson Line of the triangle with respect to P.

3. Prove that if PY in Fig. 417 is produced to cut the circum-circle at Q, then BQ is parallel to XY.

4. Prove that if P and Q are points on a circle ABC, centre S, then the Simson Lines of P and Q with respect to the △ ABC meet at an angle equal to half $\widehat{PSQ}$.

5. From the result of the previous exercise show that the Simson Lines of three points P, Q, and R with respect to a △ ABC form a triangle which is similar to the △ PQR.

6. In the triangle ABC the altitudes meet at H, and the altitude BE is produced to cut the circum-circle of △ ABC at K. Prove that HE = EK.

7. In Fig. 417, H is the ortho-centre of △ ABC, BH is produced to meet the circle ABC at K, and PK cuts AC at M and XY at L. Prove that PL = LY = LM.

8. Use the previous exercise to prove that HM is parallel to XY.

9. Prove that the line joining a point P to the ortho-centre of a △ ABC is bisected by the Simson Line of the △ ABC with respect to P. (*Hint.* Use the two previous exercises.)

Prove that O, G, and H are collinear, where O is the circum-centre of a △ ABC, G is the centroid (*i.e.*, the point of intersection of the medians), and H is the ortho-centre (*i.e.*, the point of intersection of the altitudes).

Prove also that AH = 2OA$_1$, where A$_1$ is the mid-point of BC.

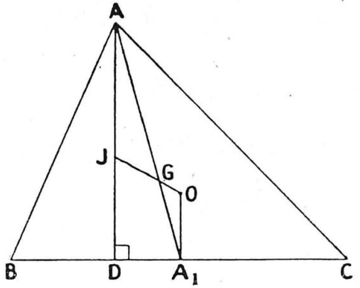


FIG. 418

Given O, G, and H the circum-centre, centroid, and ortho-centre of a △ ABC.

Prove that (i) O, G, and H are collinear.

(ii) AH = 2OA$_1$.

Proof. Produce OG to cut the altitude AD at J.
Let A$_1$ be the mid-point of BC.
Since OA$_1$ is perp^r to BC (O the circum-centre),

$$\therefore \quad AD \parallel OA_1.$$

∴ △ AJG is similar to △ A$_1$OG (equiangular).

$$\therefore \quad \frac{JG}{GO} = \frac{AG}{GA_1}.$$

But $\dfrac{AG}{GA_1} = \dfrac{2}{1}$ (G the centroid).

$$\therefore \quad \frac{JG}{GO} = \frac{2}{1}.$$

Similarly, if the altitudes BE and CF cut OG produced at J_1 and J_2, then

$$\frac{J_1G}{GO} = \frac{2}{1} \text{ and } \frac{J_2G}{GO} = \frac{2}{1}.$$

$\therefore$ J, J_1, and J_2 coincide.

$\therefore$ J, J_1, and J_2 coincide with H.

$\therefore$ O, G, and H are collinear.

Also from the similar $\triangle$s AJG and A_1OG,

$$\frac{AJ}{OA_1} = \frac{AG}{GA_1} = \frac{2}{1}.$$

$$\therefore \frac{AH}{OA_1} = \frac{2}{1}.$$

$$\therefore AH = 2OA_1.$$

Exercises: LXV

1. Prove that in Fig. 418 $AH^2 + BC^2 = 4R^2$, where R is the radius of the circum-circle.

2. Prove that in Fig. 418 the right bisector of DA_1 bisects HO. Deduce that D, E, F (the feet of the altitudes), A_1, B_1, and C_1 lie on a circle.

3. Given that N is the centre of the circle DEF, show that A_1N produced bisects AH.

4. Deduce from the previous exercise that the circle of Exercise 2 bisects AH, BH, and CH.

5. Prove that $R^2 - OH^2 = 2\,AH.HD$, where R is the radius of the circle through A, B, and C.

6. If AO is produced to cut the circum-circle at X, prove that H, A_1, and X are collinear.

The Nine-point Circle

If H is the ortho-centre and O the circum-centre of a △ ABC, the radius of whose circum-circle is *R*, prove that a circle can be drawn to pass through the feet of the altitudes, the mid-points of the sides of the △ ABC, and through the mid-points of AH, BH, and CH. Prove also that the centre of this circle is the mid-point of OH and that its radius is $\frac{R}{2}$.

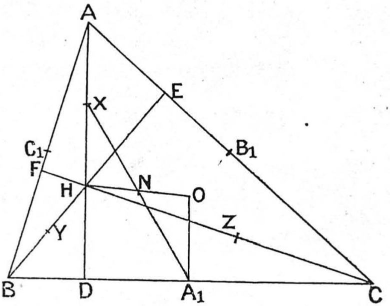


Fig. 419

Given X the mid-point of AH,

A_1 the mid-point of BC, etc.

XA_1 meets OH at N.

Prove that (i) NH = NO.

(ii) A circle with centre N and radius $\frac{R}{2}$ passes through A_1, B_1, C_1, D, E, F, X, Y, and Z.

Proof. We have AD ∥ OA_1 (both perp^r to BC).

∴ △s XHN, A_1ON, are similar (equiangular).

$$\therefore \quad \frac{HN}{NO} = \frac{XH}{A_1O}.$$

But XH = A_1O (AH = $2OA_1$).

∴ HN = NO.

Similarly, $XN = NA_1$.

Also XA_1 is the diameter of the circle through X, D, and A_1 ($\widehat{XDA_1} = 90°$).

$\therefore$ N is the centre of circle XDA_1.

Since N is the mid-point of HO,
and X is the mid-point of HA,

$$\therefore XN = \tfrac{1}{2}OA;$$

i.e., the radius of the circle $XDA_1 = \frac{R}{2}$.

Similarly, the circle with centre N and radius $\frac{R}{2}$ will pass through Y, E, and B_1 and through Z, F, and C_1.

Hence a circle with centre N and radius $\frac{R}{2}$ passes through A_1, B_1, C_1, D, E, F, X, Y, and Z.

Exercises: LXVI

1. Prove that $\triangle$s AHB, BHC, CHA, and ABC have the same nine-point circle.

2. Prove that in Fig. 418 $\widehat{DZA_1} = \widehat{ABC} \sim \widehat{ACB}$.

3. Show that if I is the in-centre and I_1, I_2, and I_3 the ex-centres of $\triangle$ ABC, then I is the ortho-centre of $\triangle I_1I_2I_3$, and the circum-circle of $\triangle$ ABC bisects I_1I_2, I_2I_3, I_3I_1, II_1, II_2, and II_3.

4. Given the base and vertical angle of a triangle, find the locus of the centre of its nine-point circle.

APPENDIX

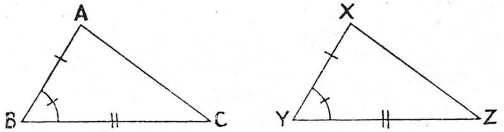
CONGRUENCE THEOREMS

APPENDIX

CONGRUENCE THEOREMS

THEOREM 2

If two triangles have two sides of the one respectively equal to two sides of the other, and the included angle of the one equal to the included angle of the other, then the triangles are congruent.



Given two $\triangle$s ABC, XYZ, in which

$$BA = YX,$$

$$BC = YZ,$$

and included angle B = included angle Y.

Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

Proof. Apply the $\triangle$ ABC to the $\triangle$ XYZ so that the point B falls on the point Y, and the side BA falls along the side YX.

Then A will fall on X (since BA = YX).

Let the triangles fall on the same side of YX.

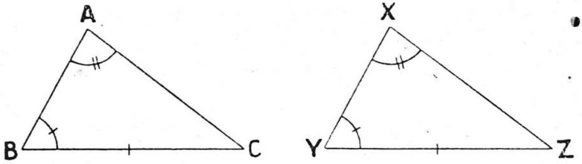
Then BC will fall along YZ (since $\widehat{B} = \widehat{Y}$), and C will fall on Z (since BC = YZ).

Hence $\triangle$ ABC coincides with $\triangle$ XYZ;

i.e., $\triangle$ ABC $\equiv$ XYZ. *Q.E.D.*

THEOREM 3

If two triangles have two angles of the one respectively equal to two angles of the other and a side of the one equal to the corresponding side of the other, then the triangles are congruent.



Given two $\triangle$s ABC, XYZ, in which

$$\hat{A} = \hat{X},$$
$$\hat{B} = \hat{Y},$$

and the side BC = the corresponding side YZ.

Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

Proof. We know that

$\hat{A} + \hat{B} + \hat{C} = \hat{X} + \hat{Y} + \hat{Z}$ (each sum equal to 180°)

and $\hat{A} + \hat{B} = \hat{X} + \hat{Y}$ (since $\hat{A} = \hat{X}$ and $\hat{B} = \hat{Y}$).

Subtracting, we get $\hat{C} = \hat{Z}$.

Apply the $\triangle$ ABC to the $\triangle$ XYZ so that B falls on Y and BC falls along YZ; then C falls on Z (since BC = YZ).

Apply the triangles on the same side of YZ;

then BA falls on YX (since $\hat{B} = \hat{Y}$)

and CA falls on ZX (since $\hat{C} = \hat{Z}$).

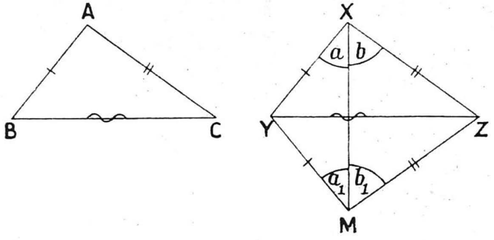
$\therefore$ A falls on X.

$\therefore$ the $\triangle$ ABC now coincides with the $\triangle$ XYZ;

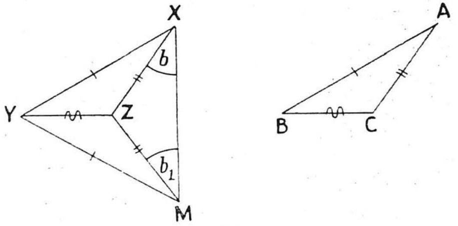
i.e., $\triangle$ ABC $\equiv$ $\triangle$ XYZ. *Q.E.D.*

THEOREM 6

If two triangles have three sides of the one respectively equal to the three sides of the other, then the triangles are congruent.



(i)



(ii)

Given two $\triangle$s ABC and XYZ in which

AB = XY,
BC = YZ,
and CA = ZX.

Denote in Fig. (ii) $\widehat{YXM}$ by a and $\widehat{YMX}$ by a_1.

Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

Proof. Apply $\triangle$ ABC to $\triangle$ XYZ, so that B falls on Y and BC falls along YZ; then C falls on Z (since BC = YZ).

Let the triangles fall on opposite sides of YZ, and let A take up the new position M. Join XM.

In Fig. (i) XM cuts YZ.

In Fig. (ii) XM cuts YZ produced.

We have XY = MY (each equal to AB).

$\therefore \quad a = a_1$ (isosceles triangle).

Similarly, $b = b_1$.

$\therefore \quad a + b = a_1 + b_1$ in Fig. (i);

i.e., $\widehat{YXZ} = \widehat{YMZ}$;

and $a - b = a_1 - b_1$ in Fig. (ii);

i.e., $\widehat{YXZ} = \widehat{YMZ}$.

Therefore in both figures

$\widehat{BAC} = \widehat{YXZ}$ (each equal to $\widehat{YMZ}$).

In the $\triangle$s BAC, YXZ,

$$\begin{cases} BA = YX & \text{(given)}, \\ AC = XZ & \text{(given)}, \\ \widehat{BAC} = \widehat{YXZ} & \text{(proved)}. \end{cases}$$

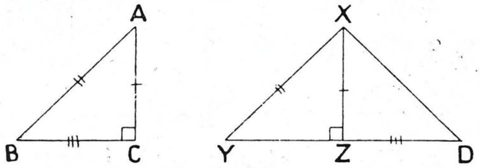
$\therefore \quad \triangle BAC \equiv \triangle YXZ$ (two sides, included angle).

Q.E.D.

NOTE. The pupil should notice that when a proof of Theorem 6 is required the correct logical sequence of Theorems 3, 4, and 6 is in the order 3, 4, and 6, and not in the order 3, 6, and 4 in which they appear in the earlier pages of the book.

THEOREM 7

If two right-angled triangles have their hypotenuses equal and one side of one triangle equal to one side of the other, then the triangles are congruent.



Given two $\triangle$s ABC, XYZ, in which

$$\widehat{ACB} = \widehat{XZY} = 90°,$$
$$AC = XZ,$$
$$\text{and } AB = XY.$$

Prove that $\triangle$ ABC $\equiv$ $\triangle$ XYZ.

Construction. Produce YZ to D, making ZD = CB. Join XD.

Proof. We know that

$$\widehat{XZY} = 90° \text{ (given).}$$

$$\therefore \quad \widehat{XZD} = 90° \quad \text{(since YZD is a straight line).}$$

In the $\triangle$s ABC, XDZ,

$$\begin{cases} AC = XZ & \text{(given),} \\ CB = ZD & \text{(const.),} \\ \widehat{ACB} = \widehat{XZD} & \text{(each } = 90°\text{).} \end{cases}$$

$$\therefore \quad \triangle ABC \equiv \triangle XDZ \quad \text{(2 sides, included angle),}$$

and, in particular,

$$AB = XD.$$

$$\text{Also} \quad AB = XY \quad \text{(given).}$$

$$\therefore \quad XD = XY.$$

$$\therefore \quad \widehat{Y} = \widehat{D} \quad (\triangle \text{ XYD is isosceles).}$$

Also $\widehat{B} = \widehat{D}$ (from the congruent $\triangle$s ABC, XDZ).

$\therefore$ $\widehat{B} = \widehat{Y}$.

In the $\triangle$s ABC, XYZ,

$$\begin{cases} BC = YZ & \text{(given),} \\ \widehat{ACB} = \widehat{XZY} & \text{(rt. angles, given),} \\ \widehat{B} = \widehat{Y} & \text{(proved).} \end{cases}$$

$\therefore$ $\triangle ABC \equiv \triangle XYZ$ (2 angles, corresponding side).

Q.E.D.

See the note on p. 373.

ANSWERS

ANSWERS

Exercises: II

2. 180°. **3.** 540°.
7. (*a*) 120°. (*b*) 60°. (*c*) 45°.
8. (*a*) 30°. (*b*) 150°. (*c*) 60°. (*d*) 72°.
9. (*a*) 90°. (*b*) $22\frac{1}{2}$°. (*c*) $112\frac{1}{2}$°. (*d*) 50°. (*e*) 150°.
10. (*a*) 45°. (*b*) 47°. (*c*) 100°. (*d*) 315°. (*e*) 230°.

Exercises: III

1. 12·8 in. **2.** 6·40 miles. **3.** 3·61 miles.
4. 2·57 miles; N. 41° W. or N. 42° W. **5.** 14·9 miles; N. 44° W.
6. 2·98 miles; N. 31° W. **7.** 11·7 ft. **8.** 47·7 ft.
9. 25·2 ft. **10.** 108 ft. **11.** 743 ft.; 801 ft.
12. 360 ft. **13.** 23°. **14.** 35·7 ft.
15. 27·7 ft. **16.** 10 ft.

Exercises: IV

1. 180°. **2.** 180°. **3.** 360°. **4.** 540°.
5. (*a*) 70°. (*b*) 36°. (*c*) 60°. **6.** (*a*) 80°. (*b*) 27°. (*c*) 120°.
7. (*a*) 160°. (*b*) 118°. (*c*) 140°. **8.** 60°. **9.** 90°. **10.** 108°.

Exercises: V

14. 82°. **15.** 541 ft. **16.** 447 yds.
17. N. 4° W. **18.** 18·9 minutes. **19.** 4·99 m.p.h.; 12·8 m.p.h.
20. 635 yds.; 166 yds. **21.** 82·1 ft. **22.** 826 ft.
23. 7·55 miles.

Exercises: VIII

1. 7·16 ft. **2.** 4·4 2ft. **3.** (*a*) 5·83 in. (*b*) 7·07 in. (*c*) 56°.
4. 14·9 in. **5.** 45°. **6.** 3·32 in.
7. (*a*) Between 14° and 15°. (*b*) 34°. (*c*) 8°. **8.** 648 ft.
9. 144 ft. **10.** 2570 yds. **11.** 48·1 ft. **12.** 22·9 ft.
14. 3·22 in. **15.** 2·78 in. **16.** 5 in.
17. (*a*) 3·57 ft. (*b*) Between 3·90 ft. and 3·91 ft.

Exercises: XI

1. (*a*) 140°. (*b*) 100°. (*c*) 62°. (*d*) $1\frac{2}{3}$ rt. angles. (*e*) $\frac{1}{2}$ rt. angle.

3. 45°. **4.** 45°. **5.** (*a*) 80°. (*b*) 10°.
6. (*a*) 60°. (*b*) 270°. (*c*) 75°.
7. (*a*) 90°. (*b*) 135°. (*c*) 135°. (*d*) 98°. (*e*) 176°.
8. (*a*) 75°. (*b*) 78°. **9.** $127\frac{1}{2}$°.
10. 144°. **12.** $x = y = 22\frac{1}{2}°$.

EXERCISES: XII

5. $l - m + n = 180°$.
6. (*a*) 140°. (*b*) 60°. (*c*) 60°. (*d*) 70°. (*e*) 110°. (*f*) 45°. (*g*) 140°.
7. $x + y = 180°$ in each case.
12. $p + q = 180°$. **13.** 50°; 130°. **14.** $x + y = 180°$.
15. (*a*) 60°. (*b*) 120°. (*c*) 80°. $l = m + n$.
16. (*a*) 70°. (*b*) 45°. (*c*) 115°. (*d*) 65°. **17.** 100°.
19. (*a*) $x + y + z = 180°$. (*b*) $y + k = 180°$.
20. (*a*) $z + m + k = 180°$. (*b*) $k + x + y = 180°$.
(*c*) $k + x + z = 180°$.

EXERCISES: XIII

4. 160°; 90°. **5.** (*a*) 60°. (*b*) $360° - x - y - z$.
6. (*a*) $(180 - 3x)°$. (*b*) $(120 - x)°$. (*c*) $(190 - 3x)°$.
7. (*a*) 30°. (*b*) 20°. **9.** (*a*) 150°. (*b*) $y = p + q + x$.
11. (*a*) 40°. (*b*) 70°. **12.** 20°. **13.** 30°.
14. 35°. **15.** 125°. **16.** 360°.
17. 40°. **18.** 360°. **19.** 540°.
23. $a + b + y = 180° + x$. **24.** $2y = 540° + x$.
25. $x = \frac{a - c}{2}$.

EXERCISES: XVI

1. 70°; 80°; 120°; 130°; 40°; 60°; 70°; 125°; 40°.
2. $x = 180° - y$. **10.** 30°.
15. 50°; 60°; 70°. **16.** 80°. **17.** 80°.

EXERCISES: XIX

1. 110°; 70°; 110°. **2.** 65°. **3.** 122°.
4. 95°. **5.** 40°. **6.** 40°; 140°; 40°; 140°.
15. 100°. **16.** 15°.

EXERCISES: XX

1. 6 in. **4.** 6 in. **5.** $1\frac{1}{2}$ in. **10.** $10\frac{2}{3}$ in.

Exercises : XXI

1. (*a*) 90°. (*b*) 140°. **2.** (*a*) 70°. (*b*) 169°.
3. (*a*) 108°. (*b*) 120°. (*c*) $128\frac{4}{7}$°. (*d*) 135°. (*e*) 156°.
4. (*a*) 72°. (*b*) 60°. (*c*) $51\frac{3}{7}$°. (*d*) 45°. (*e*) 24°. **6.** 100.
7. 105°. **8.** 30°; 30°. **9.** $48\frac{4}{7}$°; $53\frac{4}{7}$°. **12.** 120°.
14. 112°; 72°.

TEST PAPERS

Paper I

1. 12·2 miles. **2.** (*a*) 1800°. (*b*) 30°. (*c*) $2\frac{1}{2}$°.

Paper II

1. 254 yds. **3.** 93.

Paper III

2. 55°. **3.** 120°.

Paper IV

1. 17·9 miles. **2.** 107.

Paper V

1. S. 45° W. **2.** 100°.

Paper VII

1. 75°. **2.** 35°.

Paper VIII

1. 2·34 in.

Paper IX

1. Between 100 ft. and 101 ft.

Paper X

1. 12·4 cm.

Paper XI

1. 4·79 in.; 2·56 in.

Paper XII

1. 10 in.

Paper XV

1. 4·62 cm.

Paper XVI

1. 6·14 cm. 2. 144°.

Paper XVII

3. $51\frac{3}{7}°$; $51\frac{3}{7}°$; $77\frac{1}{7}°$.

Paper XVIII

1. 2 in.

Paper XIX

1. 3·49 in.

Paper XXVI

1. 24·7 ft.

Paper XXVII

1. 17·1 in.

Exercises: XXXII

1. 6·36 sq. in.
2. (*a*) 12 sq. in. (*b*) 35 sq. cm. (*c*) 6 cm. (*d*) 9·9 ft.
3. (*a*) 4 sq. in. (*b*) 28 sq. cm. (*c*) $1\frac{3}{5}$ in.
4. 39 sq. cm. 5. 22 sq. cm. 6. 8 cm.
15. 18·3 sq. in. 16. 70 sq. cm. 17. 7·16 sq. in.
18. $28\frac{1}{2}$ squares. 19. $\frac{4}{5}$ in. 20. $2\frac{1}{2}$ sq. in. 25. $\frac{1}{12}$.
41. 2081 sq. yds. 42. 2638 sq. yds.

Exercises: XXXIII

1. (*a*) 5 in. (*b*) 12 in. (*c*) 7 in. 2. 72 sq. in. 3. 48 sq. in.
4. 12 sq. in. 5. (*a*) 12 in. (*b*) 60 sq. in. (*c*) $9\frac{3}{13}$ in.
6. 1·8 in. 7. 5 cm, 8. 5 in. 9. 5 ft.
21. 7·81 cm. 22. 8·54 in. 23. 2·94. 24. 4·47 units.
28. $22\frac{1}{2}$ sq. in. 29. 13·9 cm. 30. 9·49 in.
31. (*a*) 6·63 in. (*b*) 66·6 sq. in. 32. (*a*) 8·77 in. (*b*) 10·8 in.
41. 12.

TEST PAPERS

Paper XXXI

1. 2·61 in. 3. 2·61 in.

Paper XXXII

1. 10 cm. 2. $7\frac{1}{2}$ cm.

Paper XXXIII

1. 13 cm.

Paper XXXIV

1. 68°.

Paper XXXV

3. 8 ft. 4 in.

Paper XXXVI

2. 13 ft.

Paper XXXVII

2. 4 in.

Paper XLI

1. 36 sq. in.

Paper XLIII

1. 2·13 acres.

Paper XLIV

2. 6 in.

Paper XLV

1. 21 sq. in.

Paper XLVI

1. 2.

Paper XLVII

1. 1·64 in.

Paper XLVIII

3. 2 ft.

Paper L

3. 13 ft.

Paper LIV

3. 0·708 ft.

Exercises XXXVI

1. 8 in.; 6 in.; $2\sqrt{25 - a^2}$ in.

2. 12 in.; 5 in.; $\sqrt{169 - \frac{a^2}{4}}$ in.

3. 50 in.; $17\frac{1}{2}$ in.; $2\sqrt{a^2 + 49}$ in. **4.** $1\frac{1}{4}$ ft.

5. 5 in. **6.** 0·58 in.; 8·58 in. **7.** 27 sq. in **18.** 9·49 in.

Exercises: XXXVII

1. (i) 80°. (ii) 110°. (iii) $(360 - 2x - 2y)°$ or $(2x + 2y)°$.

2. 40°. **3.** 35°. **5.** $\widehat{B} = 110°$; $\widehat{C} = 40°$. **6.** 110°.

15. (*a*) 20°. (*b*) 60°.

Exercises: XXXVIII

1. 115°. **2.** 64°. **3.** 110°.

4. 47°. **5.** 63°. **7.** 140°.

16. 55°. **17.** 115°.

Exercises: XXXIX

3. 40°. **4.** 50°.

Exercises: XL

1. (*a*) 120°; 60°. (*b*) 90°; 45°. (*c*) 72°; 36°. (*d*) 45°; $22\frac{1}{2}°$. (*e*) 36°; 18°.

2. 75°; 15°. **7.** 72°.

16. $\widehat{A} = 25\frac{5}{7}°$; $\widehat{B} = 10\frac{2}{7}°$; $\widehat{C} = 144°$. **17.** 1 : 2.

Exercises: XLII

1. 45°; 70°. **2.** 168°. **3.** 62°.

4. 60°. **5.** 46°. **6.** 70°; 140°; 40°.

7. 54°; $66\frac{1}{2}°$; $59\frac{1}{2}°$. **8.** 86°. **18.** 56°.

19. 119°. **20.** 90°.

Exercises: XLIV

1. 4 in. **2.** $4\frac{4}{5}$ in. **3.** 4·90 in.; 7·00 in.

4. 66°. **5.** 3·46 in. **6.** $4\frac{1}{2}$ in.; 5 in.; $5\frac{1}{2}$ in.

7. AB = 5 cm.; AC = 5 cm.; BC = 4 cm. **8.** 12 cm.

9. 14·1 cm. **10.** 6 cm.; 5 cm.; 2 cm. **11.** 0·83 in.; 4·83 in.

12. 1·65 cm. **30.** 4 cm.; 8 cm.; 3 cm. **31.** 1 cm.

32. 0·879 in. **33.** 30°. **34.** 12·7 cm.

Exercises: XLVII

4. 3·6 in. **6.** 1·5 in. **11.** 3·4 in.; 1·4 in.

TEST PAPERS

Paper LV

1. $121\frac{1}{2}°$.

Paper LVI

1. 110°.

Paper LVII

1. 2·48 in.

Paper LVIII

1. $3\frac{1}{2}$ in. or $8\frac{1}{2}$ in.

Paper LIX

1. 12 cm.; 4 cm. **2.** 10°.

Paper LX

1. 4·8 cm.

Paper LXI

1. 3·58 cm.

Paper LXII

1. 24 cm.

Paper LXIII

3. 35.

Paper LXIV

1. $1\frac{15}{17}$ cm. = 1·88 cm.

Paper LXVI

3. $9\frac{3}{13}$ in. = 9·23 in.

Paper LXVIII

2. 80°.

Paper LXIX

1. 4·31 in.

Paper LXXI

1. 24 cm.

Paper LXXIV

1. 26·4 in.

EXERCISES : LI

2. (i) 5 in. (ii) 11 in. (iii) 28·9 in. (iv) 1·96 in.
3. 21 sq. in. **4.** 61 sq. in. **20.** 4·2 in.

EXERCISES : LII

1. $26\frac{1}{2}$ sq. in. **2.** 18 in. **3.** 7·25 in. **4.** 7 in.
5. 8·54 in. **6.** 2·65 in. **7.** 2·125 in.

EXERCISES : LIII

1. (i) 21. (ii) 48. (iii) $\pm$ 6. (iv) $2\frac{2}{3}$. (v) $\frac{1}{4}$.
3. 2·7 in.; 4·5 in. **4.** 10·8 in.; 18 in.
5. (i) 16. (ii) 3 sq. ft. **6.** (i) $4\frac{1}{2}$. (ii) $\frac{b^2}{a}$.
7. (i) $\pm$ 12. (ii) $\pm\sqrt{ab}$. **8.** 19·2 cm.
9. 12 cm. **10.** 50 cm.
11. 1 : 2 **12.** 54°; 81°; 108°; 135°; 162°. **13.** $\frac{2}{3}$ cm.; 6 cm.

EXERCISES : LIV

1. 3 sq. in. **2.** $10\frac{1}{2}$ sq. in.
3. (i) 6 in. (ii) 1·7 in. (iii) $3\frac{6}{7}$ in. **4.** (i) 3 : 1. (ii) 8 cm.
5. $3\frac{3}{5}$ in. **6.** $3\frac{3}{5}$ sq. in.
7. 3 : 2. **8.** 1 : 2.
18. $\frac{1}{42}$.

EXERCISES : LV

1. (i) $1\frac{1}{2}$ in. (ii) $6\frac{10}{17}$ in. = 6·59 in. (iii) $7\frac{1}{2}$ in.
3. (i) $7\frac{1}{2}$ in. (ii) 14 in. (iii) 9 in.
4. $11\frac{1}{4}$ in. **5.** 3·36 in. **6.** 8·4 in. **7.** 4 sq. in.
8. 8 in. **9.** 6 sq. cm. **10.** $3\frac{1}{13}$ in. **18.** 11·9 in.

EXERCISES : LVI

1. 4·8 in.; 1·4 in. **2.** $7\frac{1}{2}$ in. **3.** 5·4 cm.; 6 cm.
4. 2 : 3. **5.** $4\frac{1}{6}$ in. **6.** 4·2 in.
7. $6\frac{2}{3}$ ft. **8.** $7\frac{1}{2}$ ft.; 41·2 sq. ft.
22. 9 in.; 5·4 in. **23.** 4·2 in. **24.** 15 cm. **25.** 1·44 in.
26. $5\frac{1}{3}$ in. **27.** $2\frac{5}{6}$ in. **29.** 12 in.

EXERCISES : LVII

1. (i) 4·8 in. (ii) 6·5 in. (iii) 2 in. or 6 in. (iv) 14 in. (v) 6 in.
2. (i) 0·2 in. (ii) 9 in. (iii) 2 in.

3. (i) 7·2 in. (ii) 2·4 in. (iii) 2·5 in. (iv) 12 in.
7. $3\frac{1}{8}$ in. **8.** 3 in.; 8 in. **9.** 4·36 ft. **10.** 8·7 miles.
11. 36·9 miles **19.** 300 ft. **20.** 24 ft. **21.** 38·15 ft.

Exercises: LVIII

1. 5 sq. in. **2.** 12·3 cm. **3.** 1 : 330. **4.** $5\frac{1}{3}$ sq. in.
5. 4 : 9. **6.** 14 cm. **7.** 12 : 25. **8.** 25·6 in.
9. $7\frac{1}{2}$ cm. **10.** 3·84 sq. in. **11.** 1 : 2. **12.** 9 : 100.
13. 49 : 52. **16.** $\frac{7}{9}$.

TEST PAPERS

Paper LXXVII

1. (*a*) $2\frac{1}{7}$ in.; $2\frac{6}{7}$ in. (*b*) $3\frac{1}{3}$ in.; $8\frac{1}{3}$ in.
3. 6 in. and 14 in.; $17\frac{1}{2}$ in. and $10\frac{1}{2}$ in.; 7 in. and 5 in.

Paper LXXVIII

1. 5 in.

Paper LXXIX

1. $22\frac{1}{2}$ in.; 64 : 225.

Paper LXXX

1. 10·3 cm.

Paper LXXXI

2. 8 : 3.

Paper LXXXII

1. $34\frac{2}{7}$ in. **2.** $6\frac{2}{3}$ in.; 9 : 25.

Paper LXXXIII

2. 4 : 1.

Paper LXXXIV

1. (i) 6·93 cm. (ii) $21\frac{1}{3}$ cm. (iii) 8·4 cm.
2. 4 ft. $11\frac{3}{4}$ in. **3.** 1 : 9.

Paper LXXXV

1. 9 : 25; $2\frac{2}{5}$ in.; $2\frac{1}{2}$ in.

Paper LXXXIX

2. (i) 3 : 1. (ii) 3 : 1; 1 : 1.

Paper XC

1. $6\frac{3}{5}$ in.

Paper XCII

1. 2 : 9.

Exercises: LXII

1. 2 : 3.